An Introduction to Knot Theory

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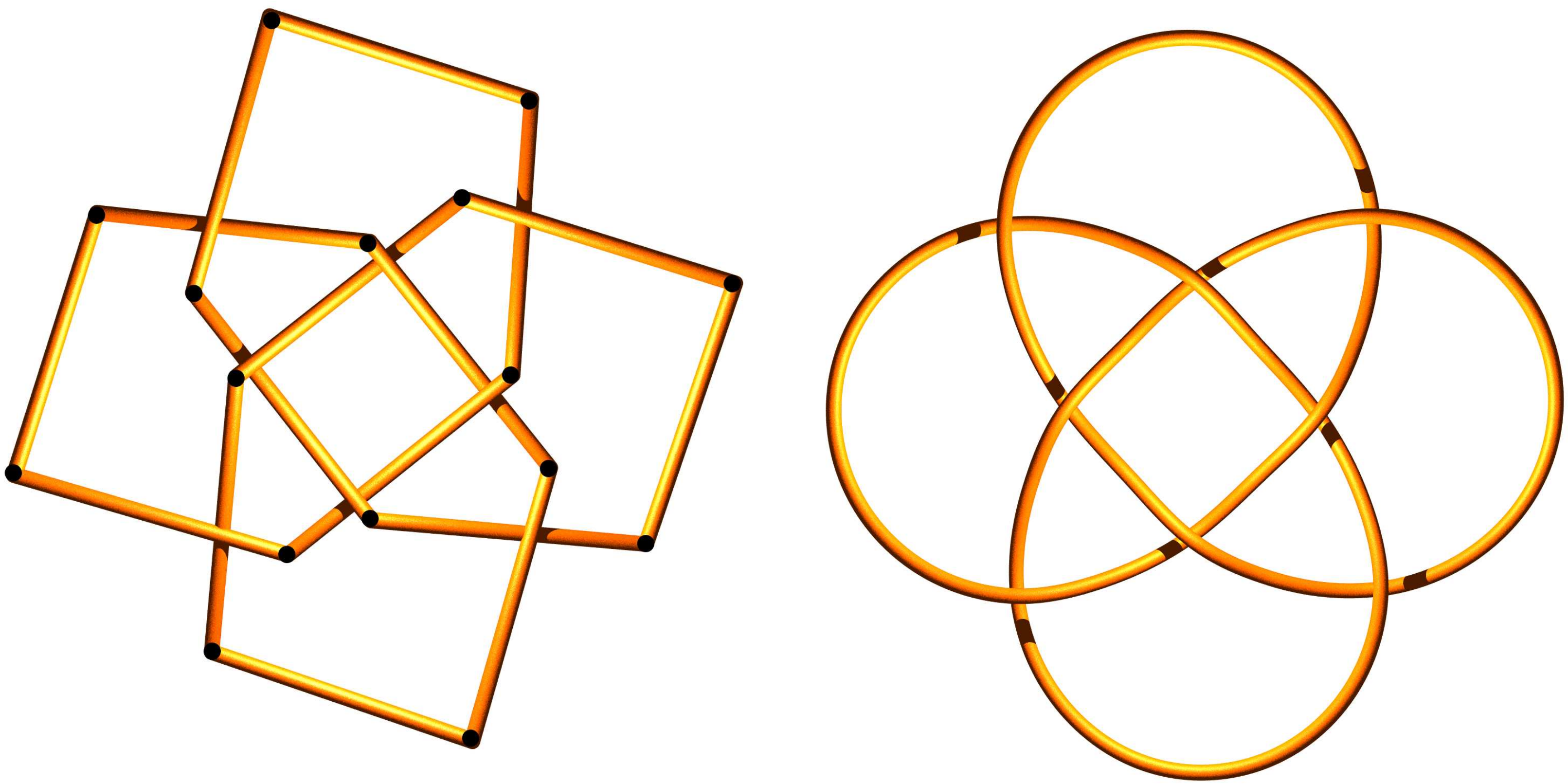
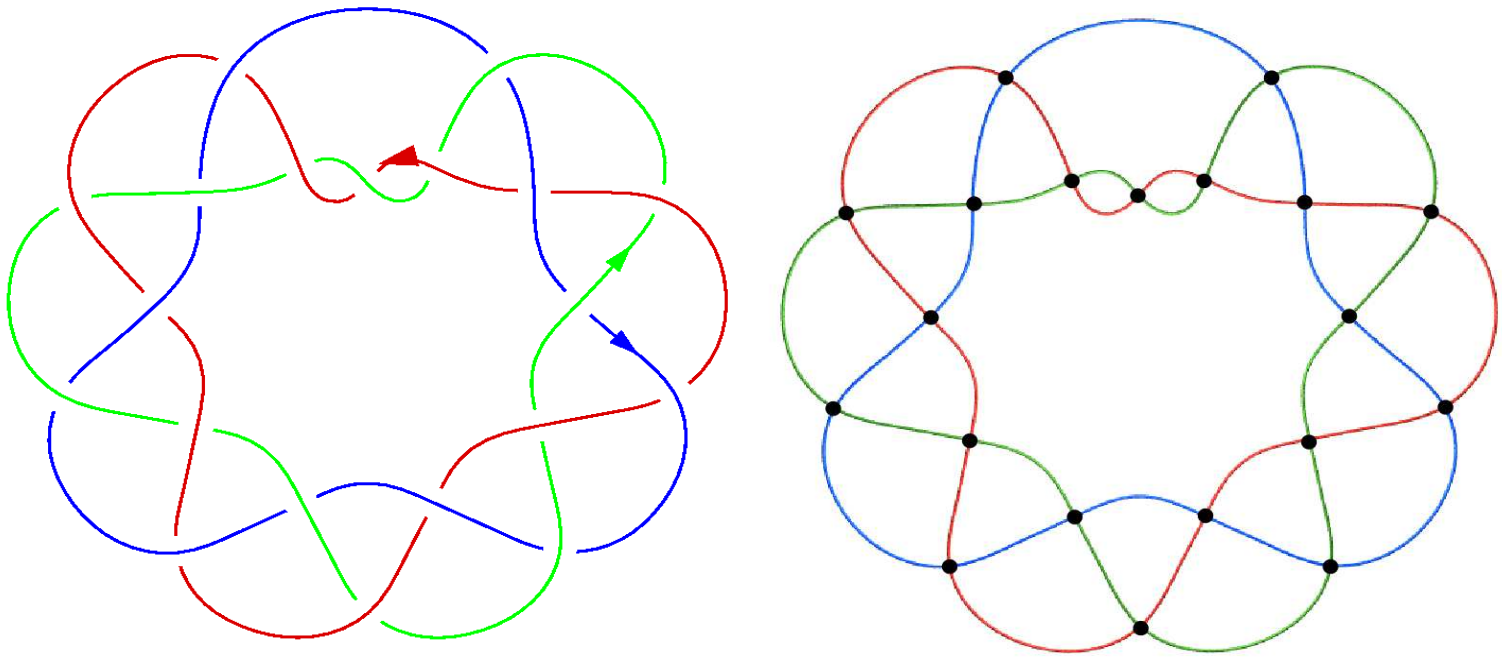
Preliminaries

- A link L of m components is a subset of S^3 that consists of m disjoint, piecewise linear simple closed curves, each of which is said to be a component of the link. A knot has one component.



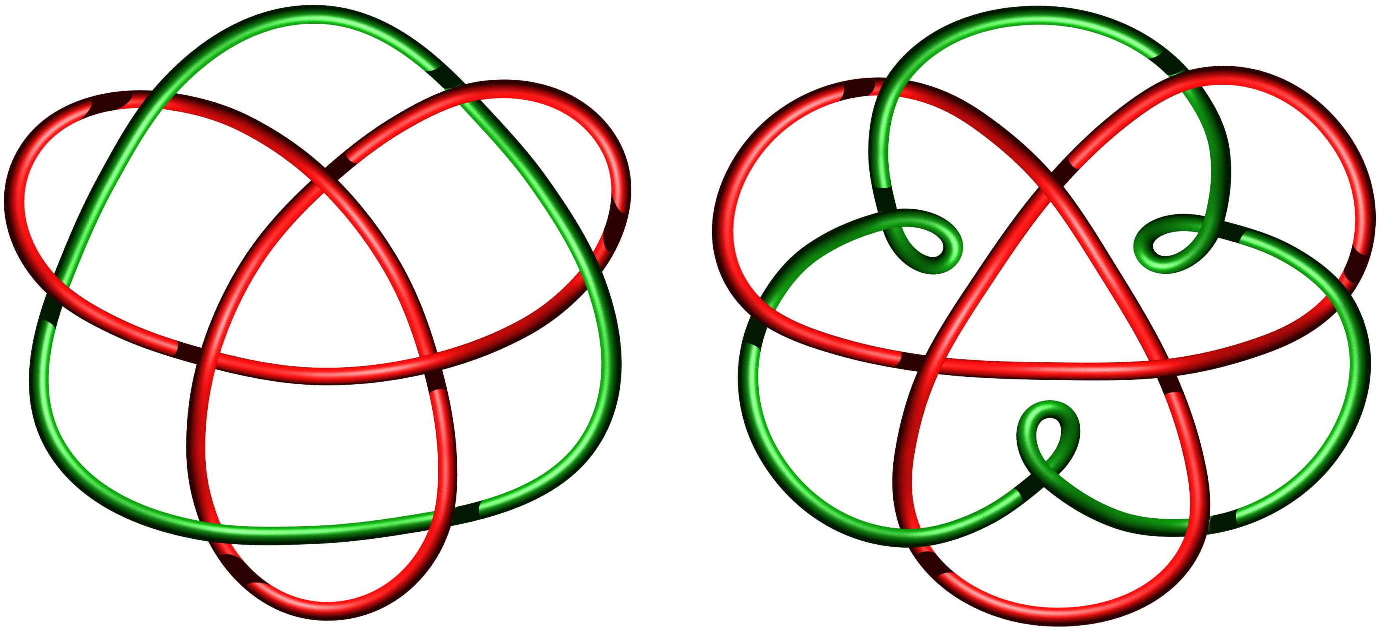
An 8 crossing knot

- A diagram D of a link L is the projection onto a plane (or 2-sphere) of L such that every point of D is at most two points from L. If a point of D is two points from L then it is said to be a crossing of D, called a component crossing if both points belong to the same component of L, otherwise it is called a link crossing.
- Links L_1 and L_2 are said to be topologically equivalent if one can be transformed into the other using a smooth collision free motion. We say the diagrams for L_1 and L_2 are equivalent if the links are equivalent.
- It is conventional to indicate on a link diagram the additional information required to construct a link equivalent to the one from which the diagram was made. This information takes the form of over/under passes, indicated by a break in the image of the underpassing arc in a neighbourhood of the crossing.



- We shall treat a link diagram as a 4-regular graph on S^2 by considering each crossing as a vertex of the graph and the portion of the curve between two consecutive crossings as an edge between the two vertices.
- In a link diagram, a circuit in which no two adjacent edges appear consecutively in the circuit will visit all vertices in some component and no others, and we shall call such a circuit a traversal of that component

- The traversal is said to be alternating if consecutive crossings alternate between underpass and overpass. A link traversal is a set of component traversals containing exactly one component traversal of each component.
- An alternating diagram is a link diagram for which every link traversal is alternating and an alternating link is one for which at least one diagram is alternating.
- For a link L, we consider the set S of all link diagrams of all links equivalent to L. A diagram of fewest crossings in S is called a minimal diagram for L.



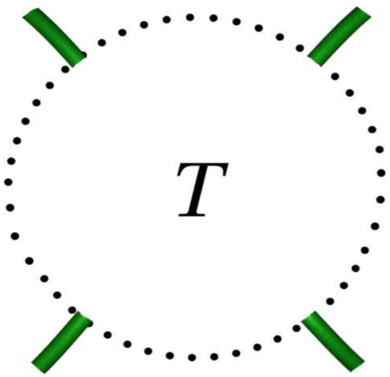
Two configurations for a link

Knot theory and Nature

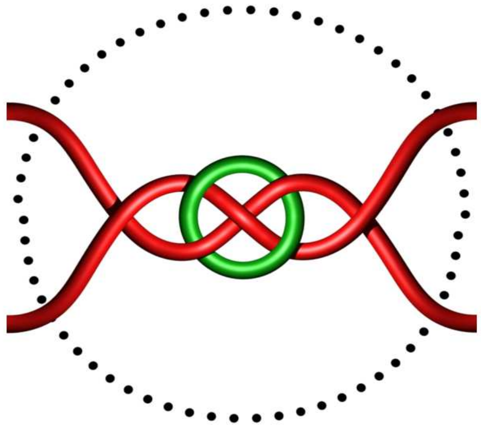
- Polymers routinely become knotted.
- For proteins, knot theory is likely to be useful, not because proteins are or are not knots, but because the geometry and motions of protein backbones may be modelled using techniques from knot theory.
  Two proteins or subsegments of proteins are similar if there is a motion that transforms one into the other while avoiding backbone self-collisions.
  Knot invariants help assess the similarity.
- To understand the supercoiling and knotting behaviours observed in DNA, in particular DNA packing and unpacking by topoisomerases.
  Wasserman, Cozzarelli and company have predicted that certain enzymes would knot DNA molecules.
  Site specific recombination involves topological changes in the DNA substrate.

- Measuring changes in crossing number have also been instrumental in understanding the termination of DNA replication and the role of enzymes in recombination.
- Viruses can knot DNA.
- To characterize the process of gene descrambling in ciliates by forming a short sequence of knots via simple surgeries.
- Chirality of catananes etc. are important to chemists.
- The periodic orbits of the Lorenz attractor define knots.
- Knot invariants appear in String theory.
- All fabrics are knotted and tying your shoes makes a trefoil.

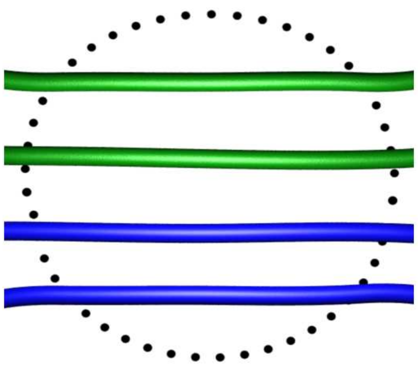
Tangles and Groups



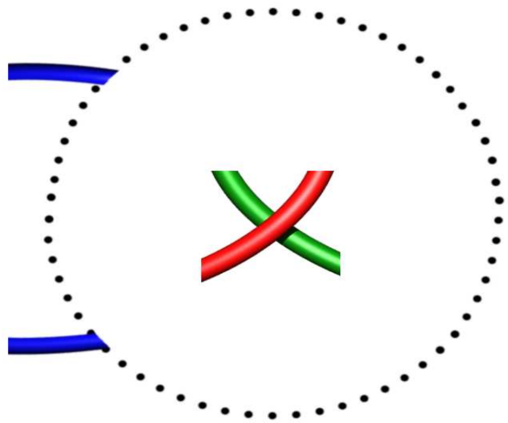
A 4-tangle



Non-trivial 4-tangle



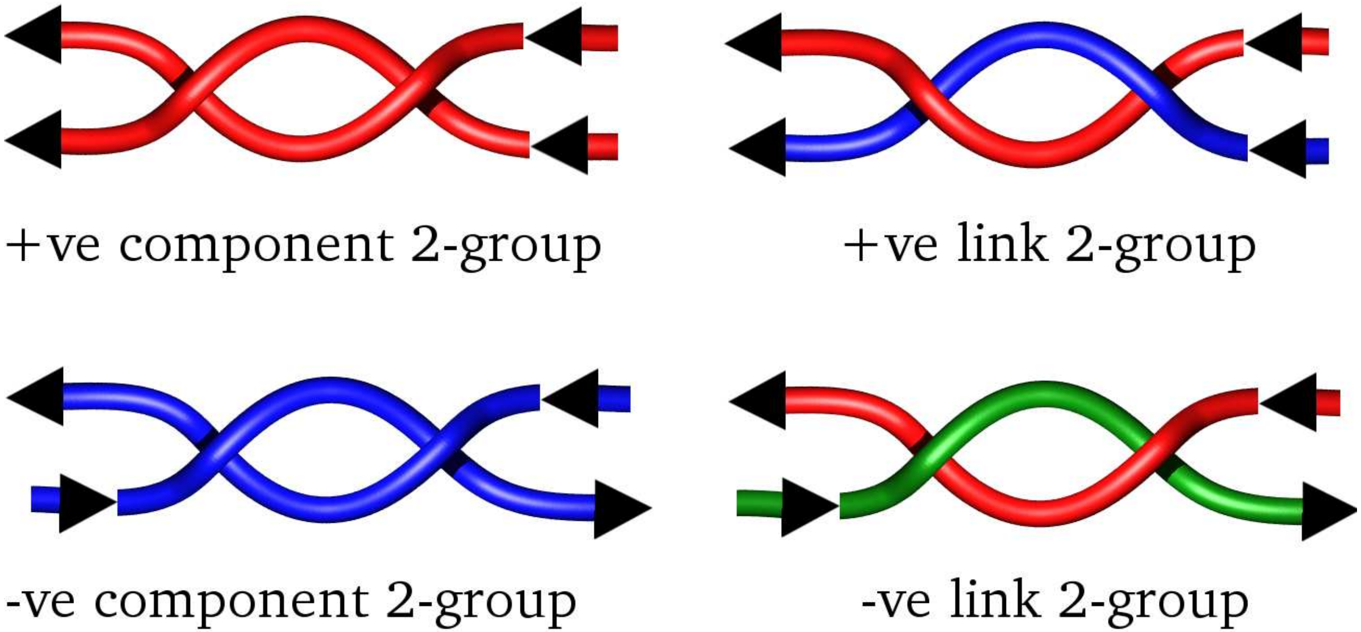
Trivial 8-tangle



Non-trivial 2-tangle

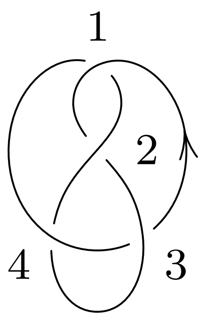
- John Horton Conway coined the term tangle in 1969.

Groups



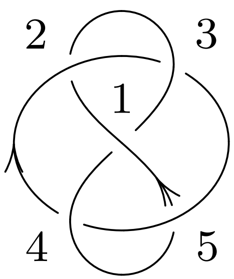
- A group of size one (a crossing) is called a loner and is not signed.

Group Codes



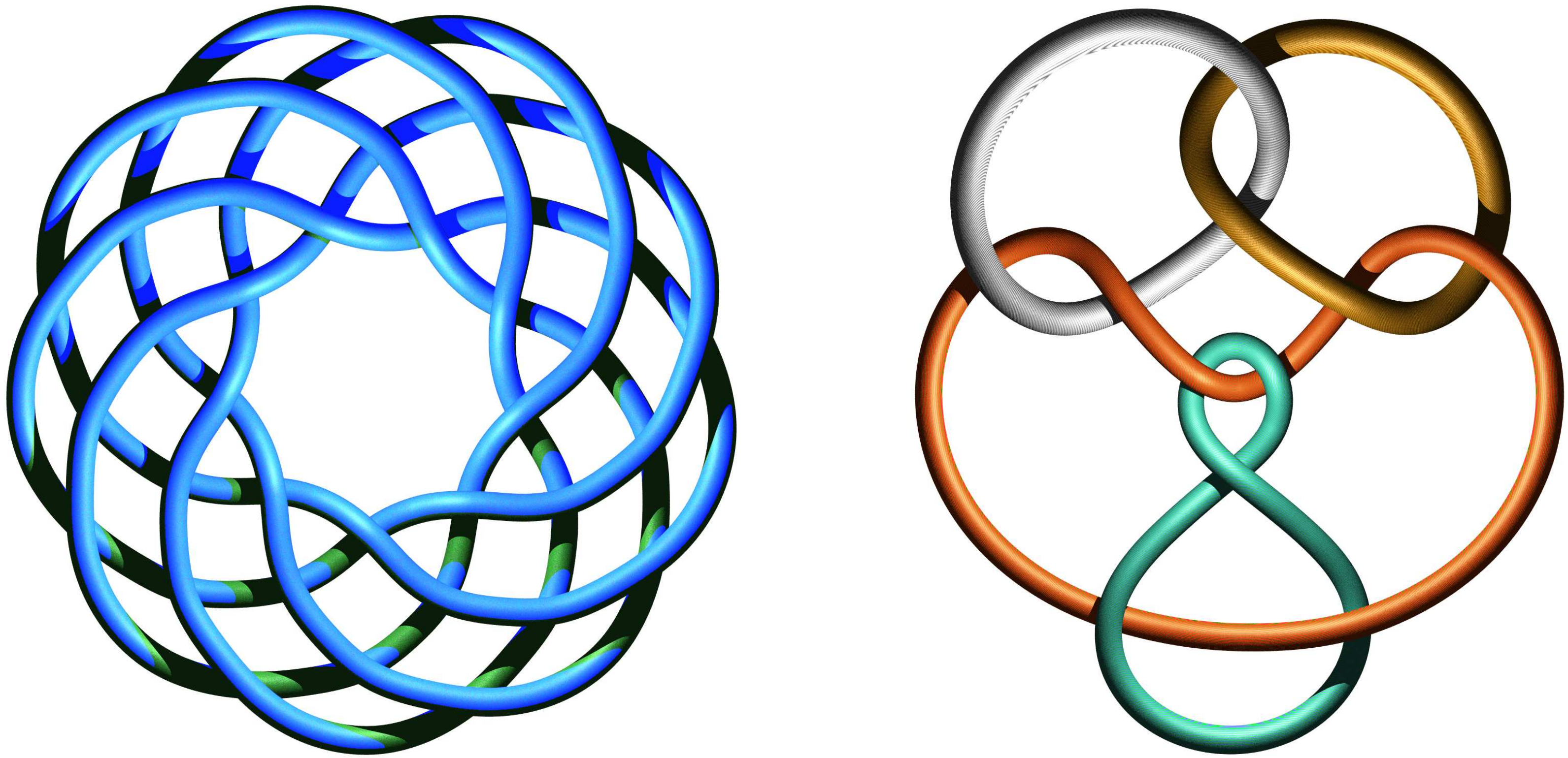
$$\text{Gauss code:}\quad \overset{+}{1},\overset{-}{2},\quad \overset{+}{3},\overset{-}{4},\quad \overset{+}{2},\overset{-}{1},\quad \overset{+}{4},\overset{-}{3}$$

$$\text{Group code:}\quad \underbrace{1,2,}_{-2_1\,,}\quad \underbrace{3,4,}_{-2_2\,,}\quad \underbrace{2,1,}_{-2_1\,,}\quad \underbrace{4,3}_{-2_2}$$



$$\text{Gauss code:}\quad \overset{+}{1},\quad \overset{-}{2},\overset{+}{3},\quad \overset{-}{1},\quad \overset{+}{4},\overset{-}{5}\quad :\quad \overset{+}{2},\overset{-}{3},\quad \overset{+}{5},\overset{-}{4}$$

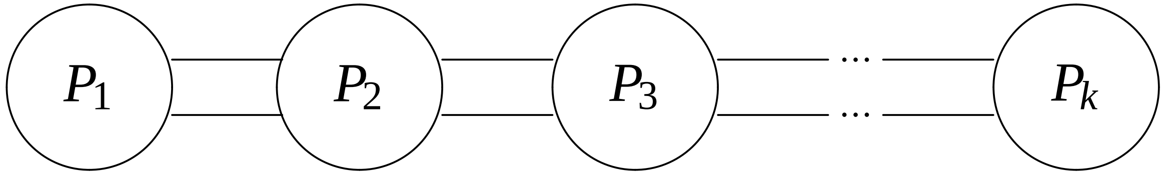
$$\text{Group code:}\quad \underbrace{1,}_{1_1\,,}\quad \underbrace{2,3,}_{2_1\,,}\quad \underbrace{1,}_{1_1\,,}\quad \underbrace{4,5}_{-2_2}\quad :\quad \underbrace{2,3,}_{2_1\,,}\quad \underbrace{5,4}_{-2_2}$$

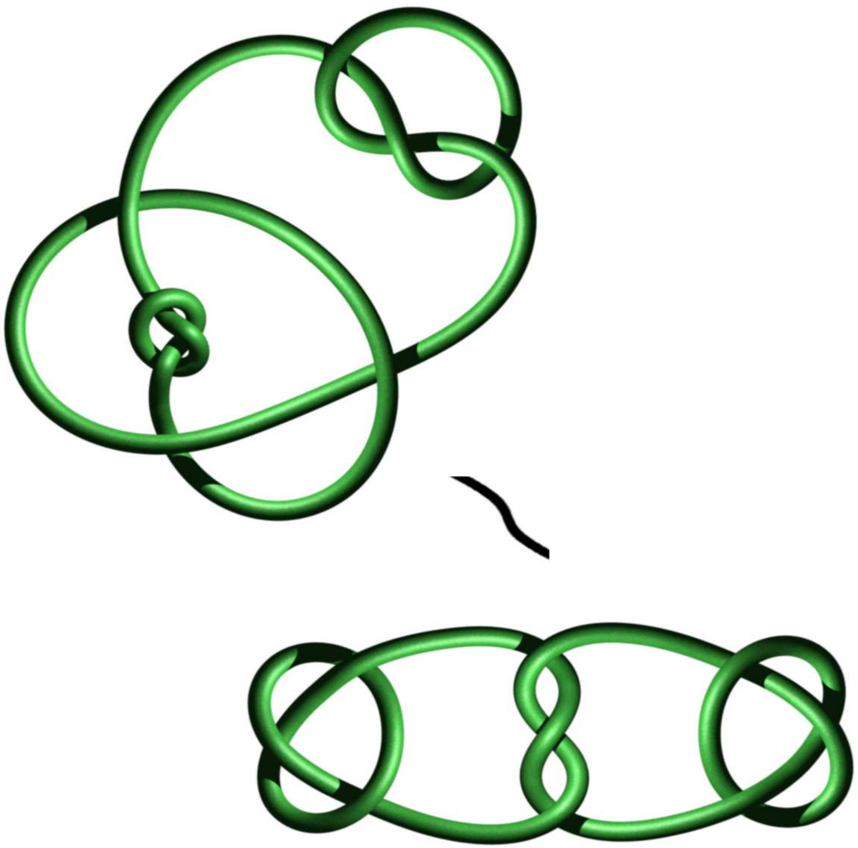


Non-alternating prime links

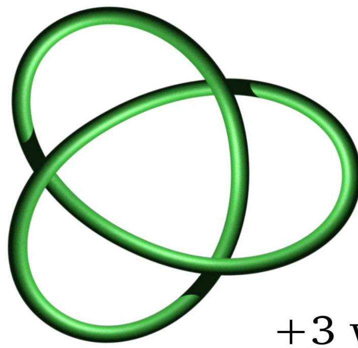
- A prime link has no minimal diagram with a 2-tangle.
- A composite link is a sum of two or more prime links and has at least one minimal diagram with at least two 2-tangles.

- Notation: $P_1 \ \# \ P_2 \ \# \ P_3 \ \# \ \ldots \ \# \ P_k$

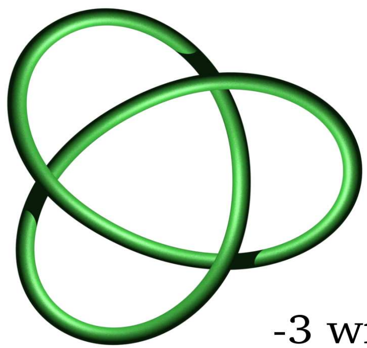




$K_3 \ \# \ K_3 \ \# \ K_3$

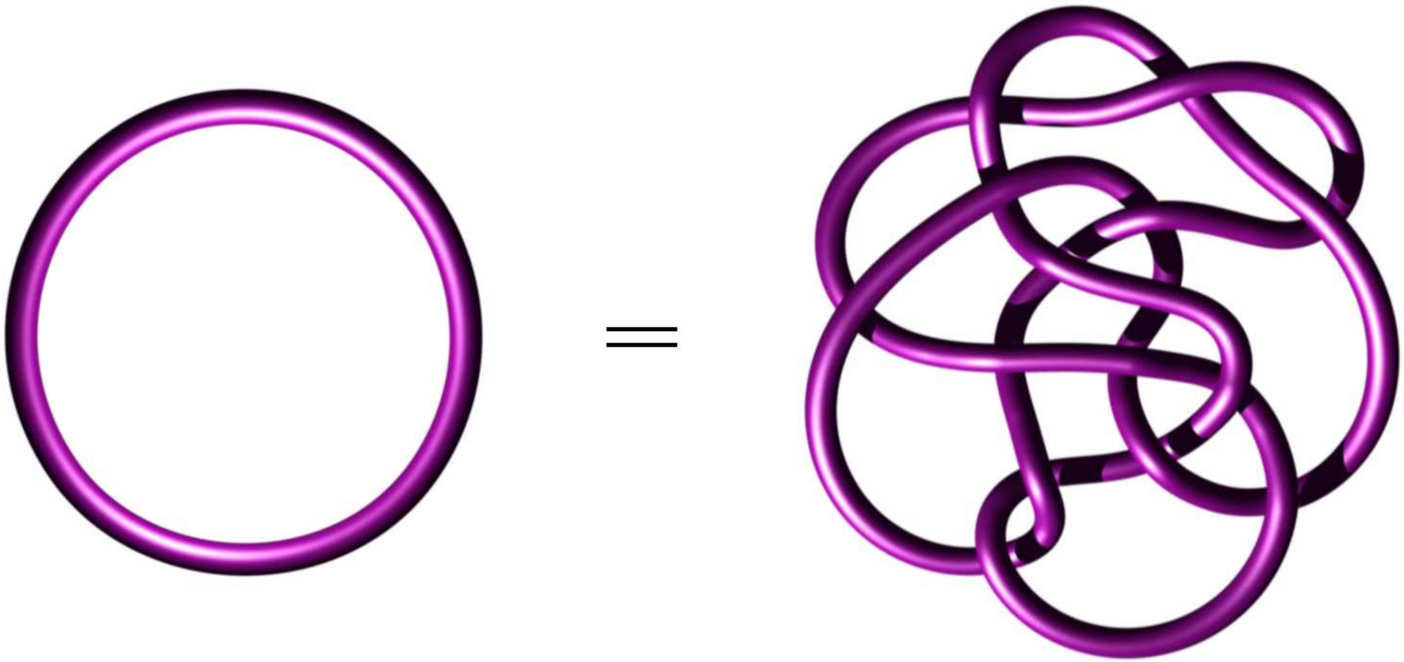


+3 writhe



-3 writhe

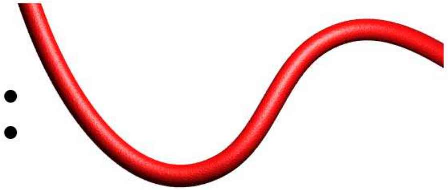
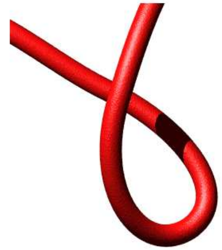
Knotted or Unknotted?

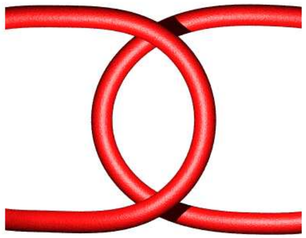
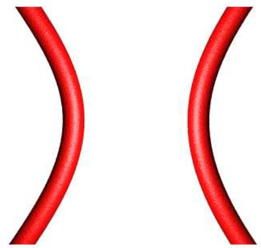


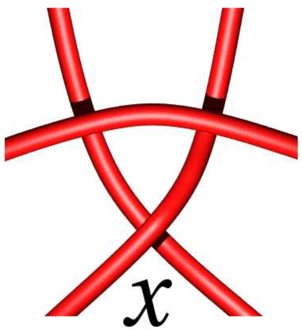
The Unknot

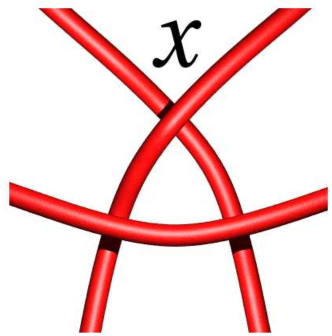
- To determine knottedness is one of the central problems in Knot theory.

Reidemeister Moves

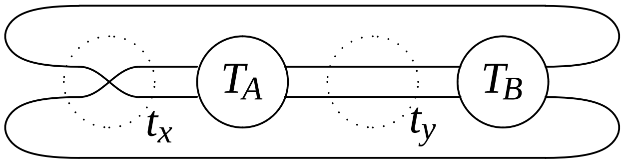
R1:  $\sim$ 

R2:  $\sim$ 

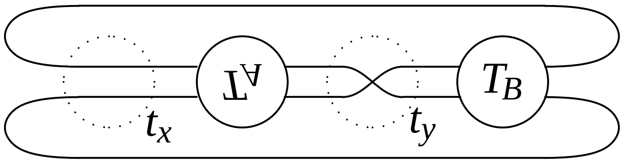
R3: 

$\sim$ 

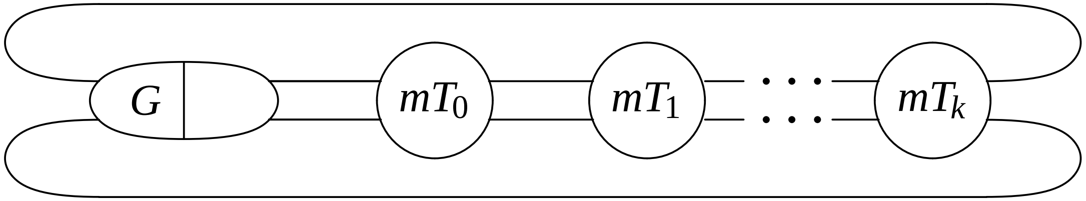
Tait's Flyping Conjecture



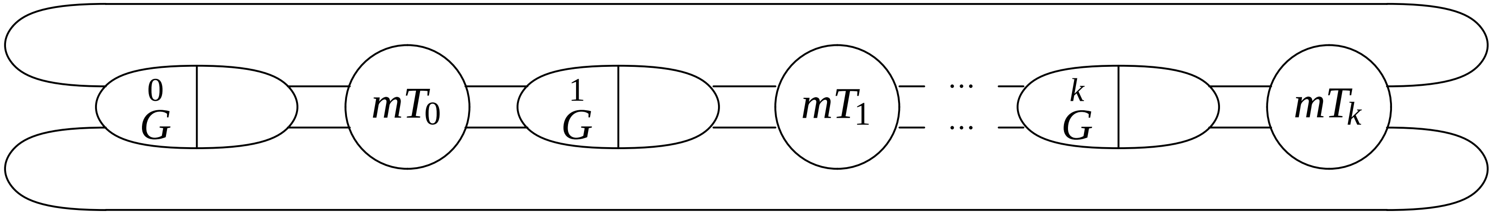
Before flype move



After flype move

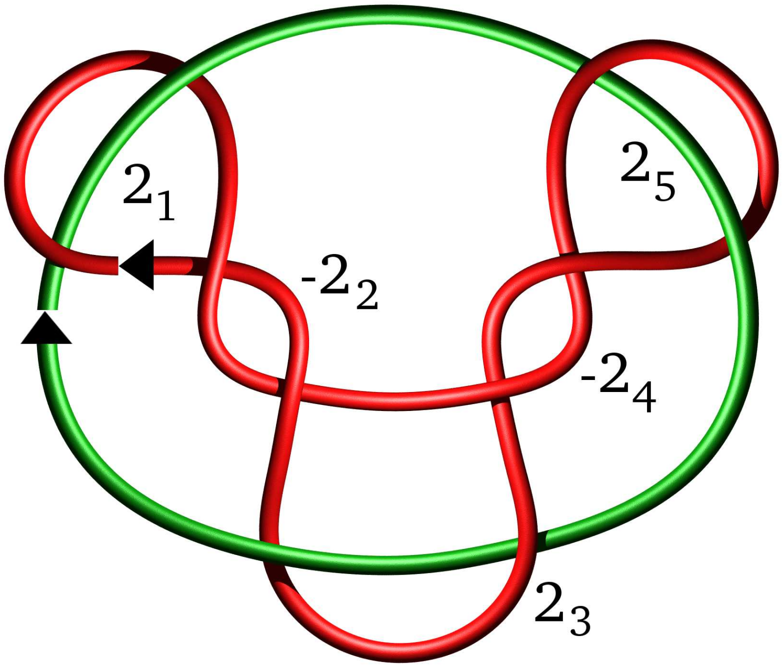


The Flype orbit for group G

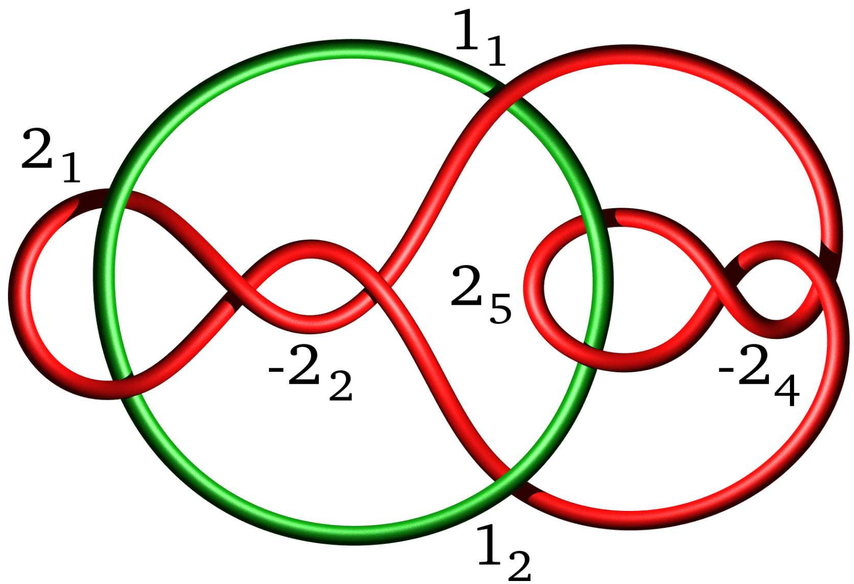


What the Master Array sees for G

- Two configurations for a 10 crossing 2-component link.



a) Full group

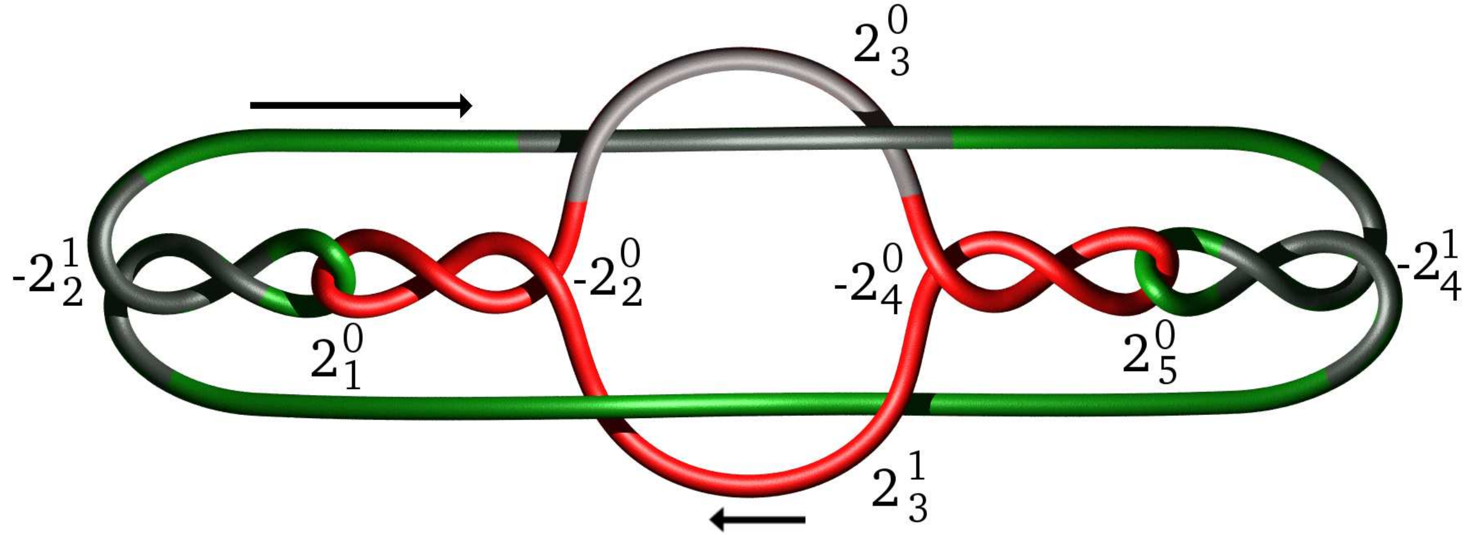


b) Split group

A Group code for a) $2_1, -2_2, -2_4, 2_5, -2_4, 2_3, -2_2 : 2_1, 2_5, 2_3$

A Group code for b) $2_1, -2_2, 1_1, -2_4, 2_5, -2_4, 1_2, -2_2 : 2_1, 1_1, 2_5, 1_2$

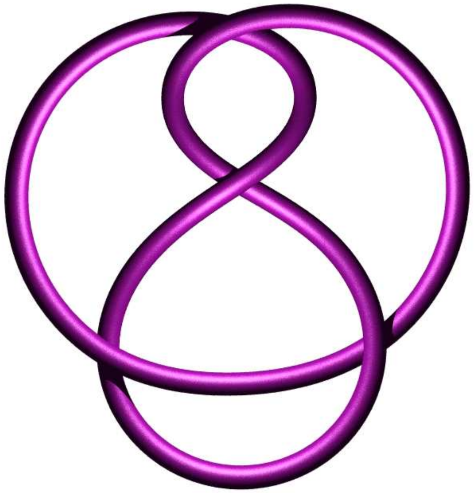
The Master Array



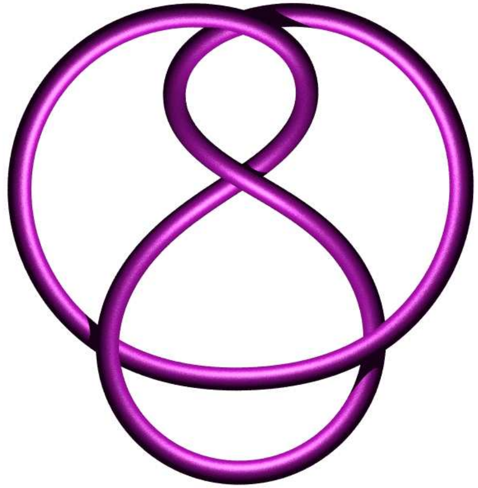
$$2_1^0, -2_2^0, 2_3^0, -2_4^0, 2_5^0, -2_4^0, 2_3^1, -2_2^0 : 2_1^0, -2_2^1, 2_3^0, -2_4^1, 2_5^0, -2_4^1, 2_3^1, -2_2^1$$

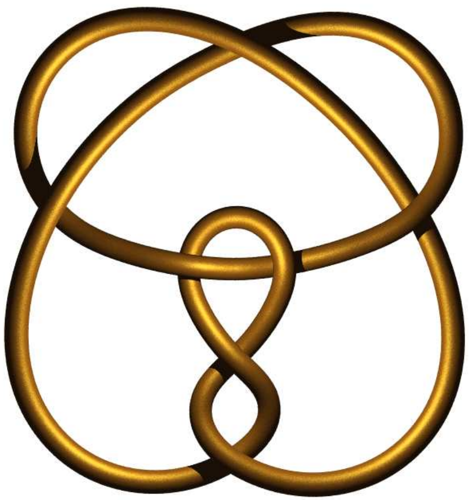
- The Flyping Conjecture.
- The non-interference of flype orbits.

Amphicheiral and Chiral

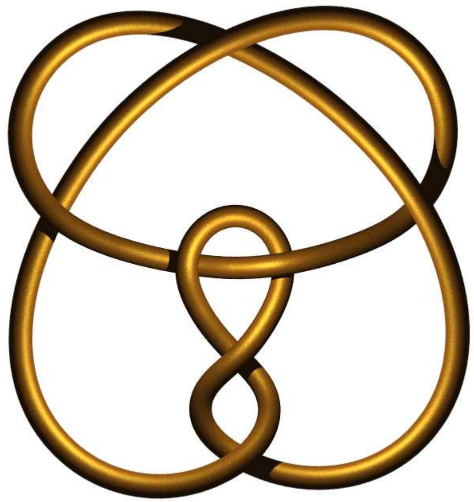


$=$





$\neq$



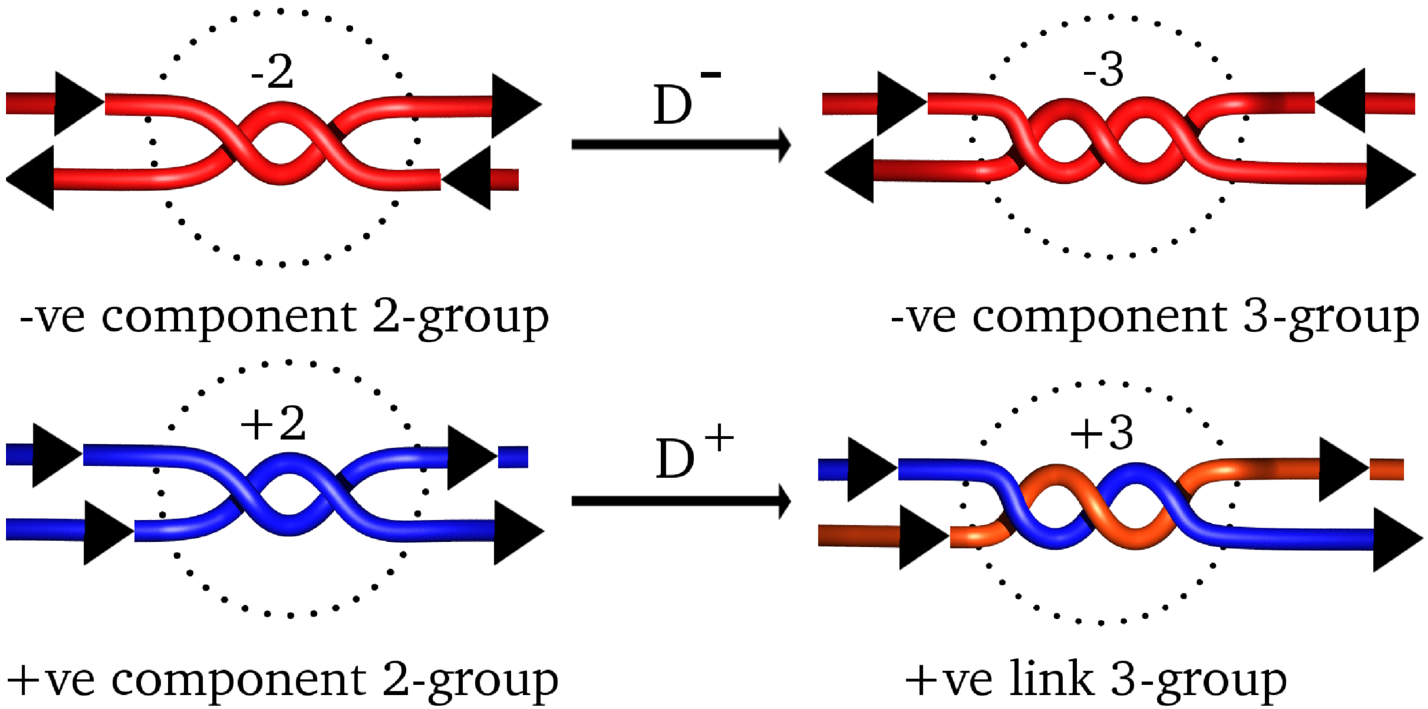
Writhe

- Given an orientation on each component of a link diagram D, the *writhe* of a crossing is either $+1$ or -1.

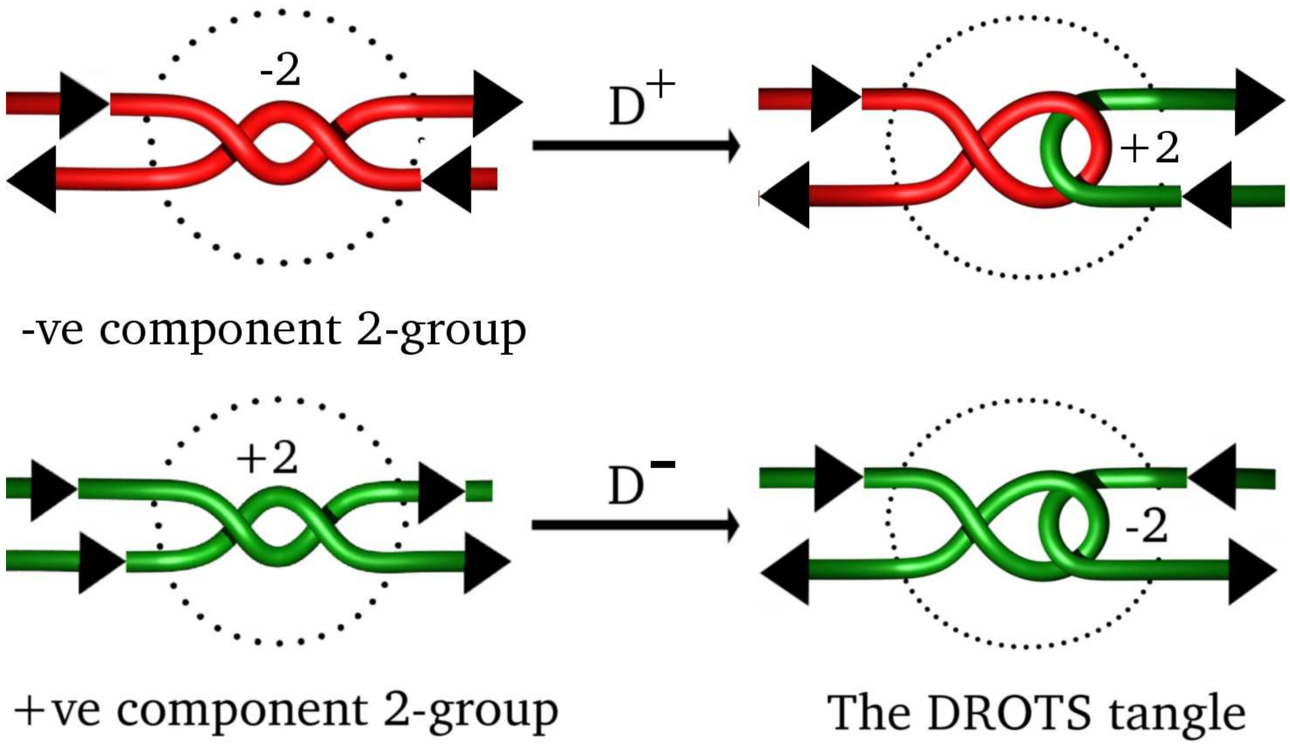
$+1$ writhe $\qquad\qquad$ -1 writhe

- With the writhe information as part of the Master Array then it's a complete invariant that distinguishes a link from its mirror if the link is chiral.
  That is, we will obtain one Master Array for a chiral link and a different Master Array for its mirror and we will only produce one Master Array for an amphicheiral link.
- We can produce such a complete invariant in linear time but may lose information about the link's symmetry group.

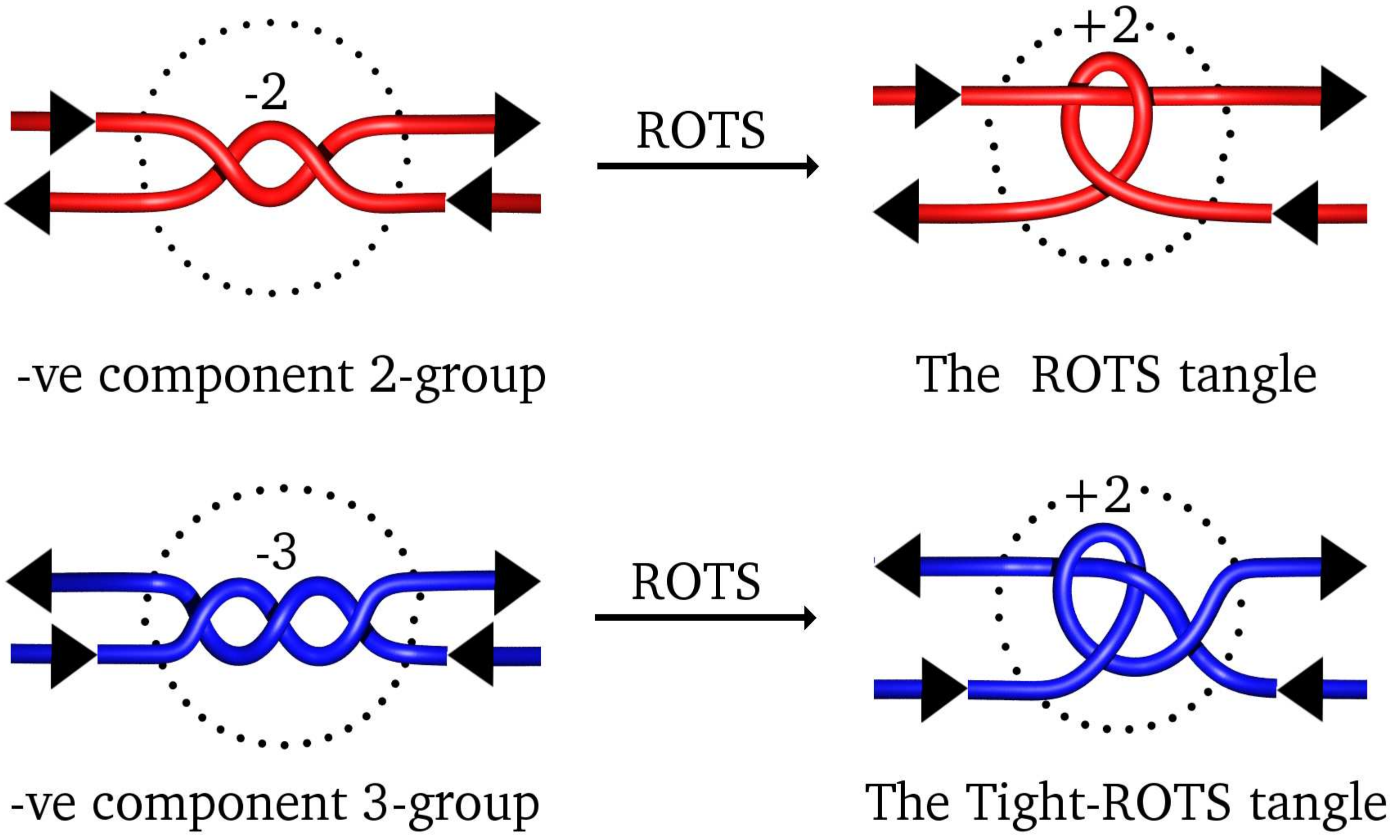
The *D* surgery



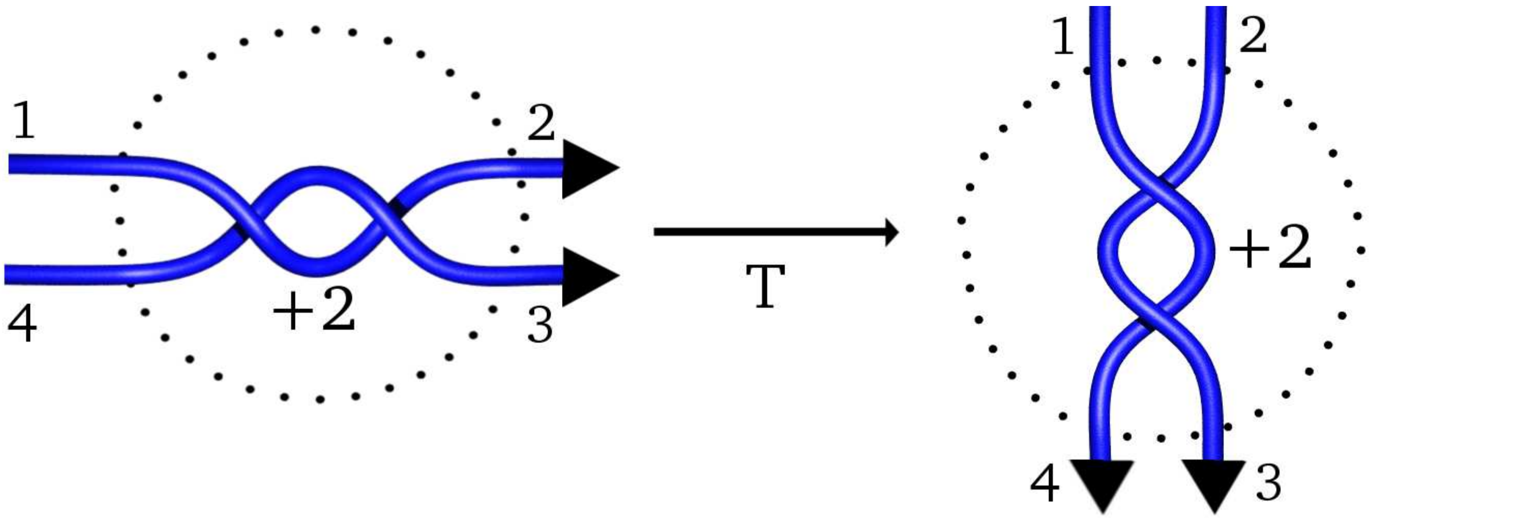
The D surgery

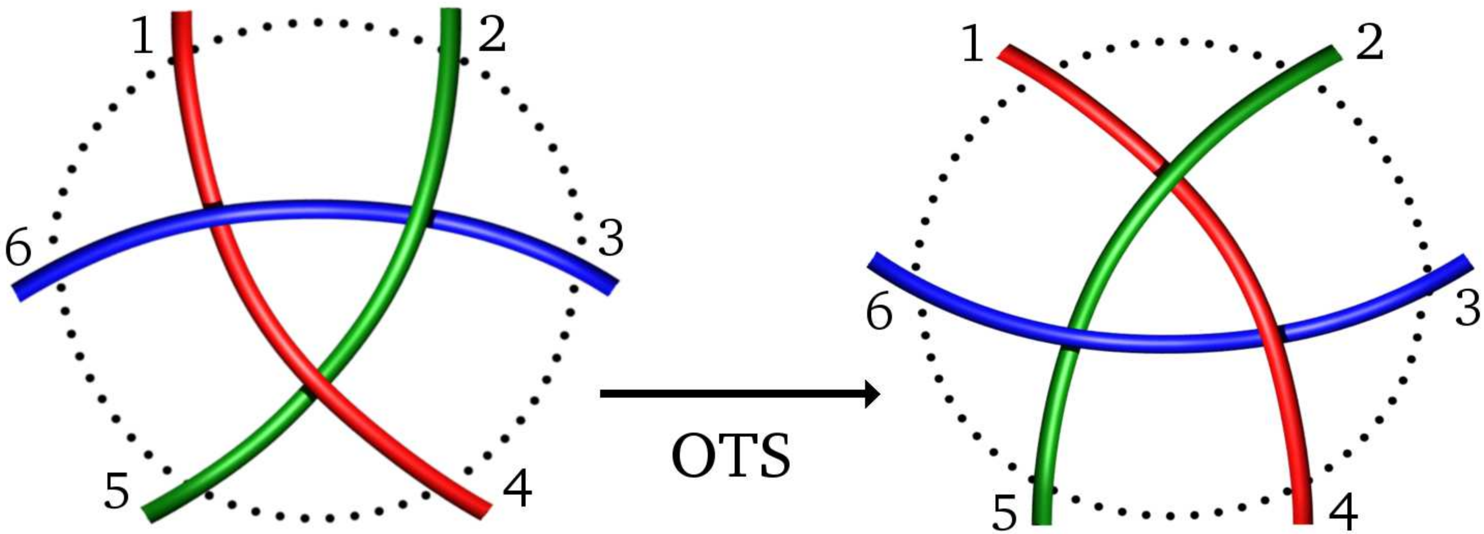


The *ROTS* surgery

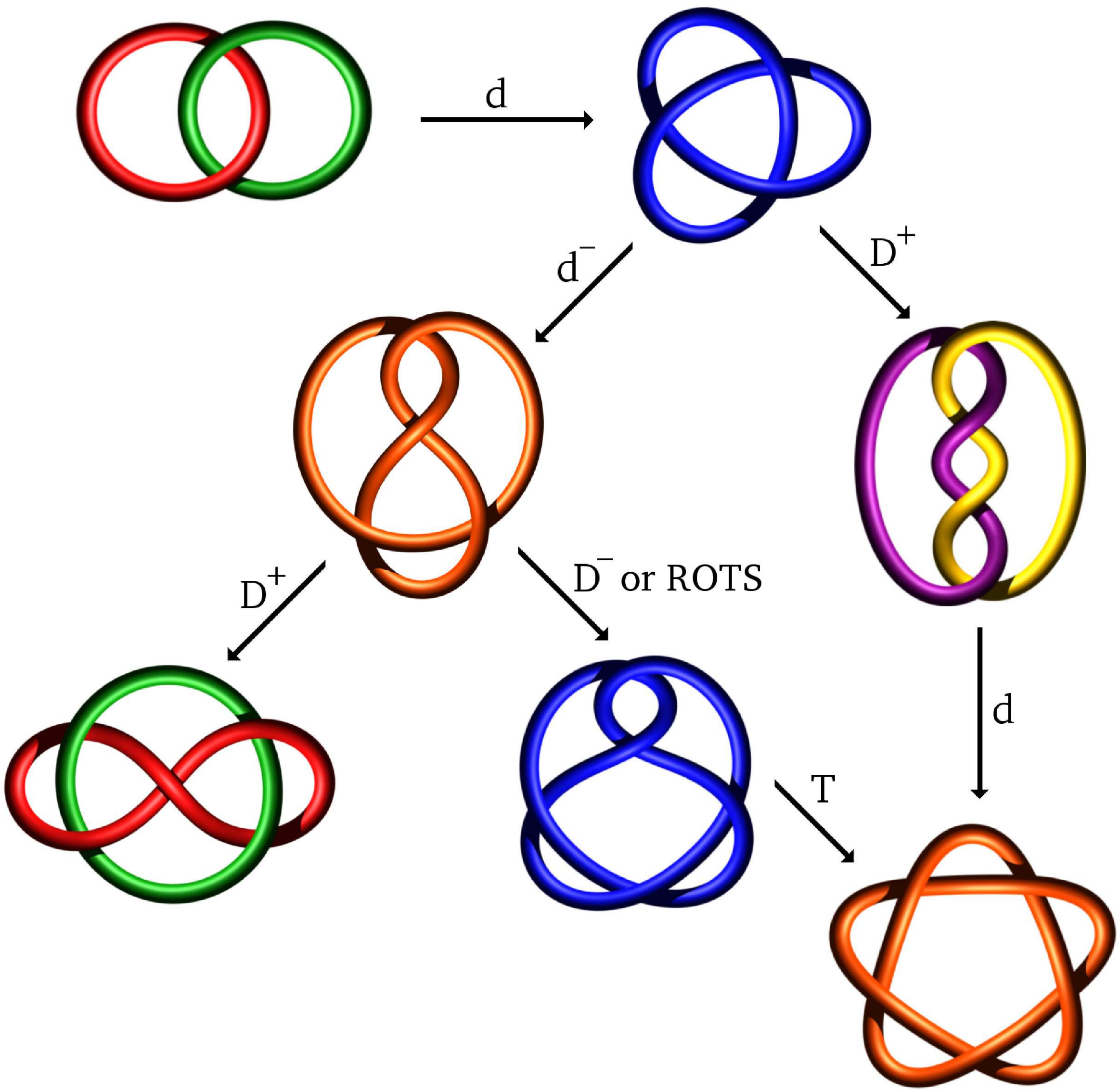


The T and OTS surgeries





Prime Alternating Link Enumeration



Two Enumeration Schemes

- $n \to n+1$ (bottom up).
  We start with the four crossing figure-eight knot and the torus link.
  With the complete collection of links at n crossings and $k=1$ (the knots) we apply D^- to all applicable component groups (including loners) and for a selected subcollection we apply $ROTS$ to negative 2 and 3-groups. Recall that for positive groups, we apply D^- to 2-groups only. We obtain roughly 98% of the knots at $n+1$ crossings. Then from a small subcollection of these $n+1$ knots we apply T to positive 2-groups once. Then from a very small sub-collection of these $n+1$ knots obtained by one application of T we apply OTS. With the knots from the first OTS output, we apply OTS iteratively until no new knot is made. To enumerate the $n+1$ crossing links with $k \geq 2$ components, we apply D^+ to all applicable component groups (including loners) on the collection of n crossing links with $k-1$ components.

We do as we did for knots above but with a very small sub-collection of the n crossing links with k components to avoid any redundancy. That is, the operators defer to one another. We obtain again roughly 98% of the links from the D^- and D^+ and $ROTS$ outputs and we apply T and OTS as described above for the complete collection of links with $n+1$ crossings and k components. Finally, when $k = \lfloor n/2 \rfloor$ we have enumerated all the links at $n+1$ crossings.

- $n \longrightarrow n$
  We proved in 2002 that there exists a finite sequence of T and OTS operations to obtain a link with n crossings starting with any other link with n crossings. Thus, beginning with the simplest link with n crossings, namely the $(n,2)$ torus link which consists of one group, we apply T to any group (or subgroup) and OTS iteratively to obtain the complete set of prime alternating links on n crossings.

- The 24 crossing (unoriented) prime alternating links were enumerated distributed over 40 desktops in 40 hours on September 29th and 30th, 2007. Coding by Bruce Fontaine.

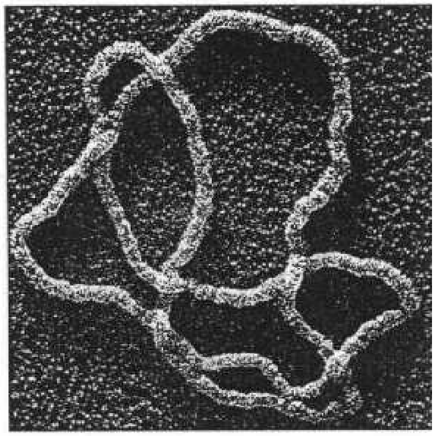
  At 1 Components: 128,564,665,125
  At 2 Components: 170,407,462,900
  At 3 Components: 89,132,658,209
  At 4 Components: 24,737,359,787
  At 5 Components: 4,078,666,332
  At 6 Components: 425,398,302
  At 7 Components: 29,712,652
  At 8 Components: 1,469,675
  At 9 Components: 53,617
  At 10 Components: 1,429
  At 11 Components: 29
  At 12 Components: 1

  A total of 417,377,448,058.

Knotilus

- There are 100 billion prime alternating links archived.
- Can enter any gauss code for any link to obtain a diagram.
- Can download the java applet to draw and anneal with full control, any link and a gauss code is constructed.
- Can do all face inversions for the diagram.
- Writhe is given for the diagram.
- One click to produce the mirror image.
- Knotilus provides by drawing or submitting a code for an alternating diagram the following:
  a) A complete invariant (the Master array).
  b) The number of orientations.
  c) The invertibility of a knot.
  d) If the link is unoriented amphicheiral or chiral.
  e) How many of the link's orientations are oriented amphicheiral and a link to one such orientation.
  f) A cycle decomposition on the vertices (or sometimes the edges) for each symmetry of the symmetry group for the link which can be submitted to GAP.
  g) The name of the symmetry group for a knot.
- Outputs various formats including Povray.

Knotted DNA



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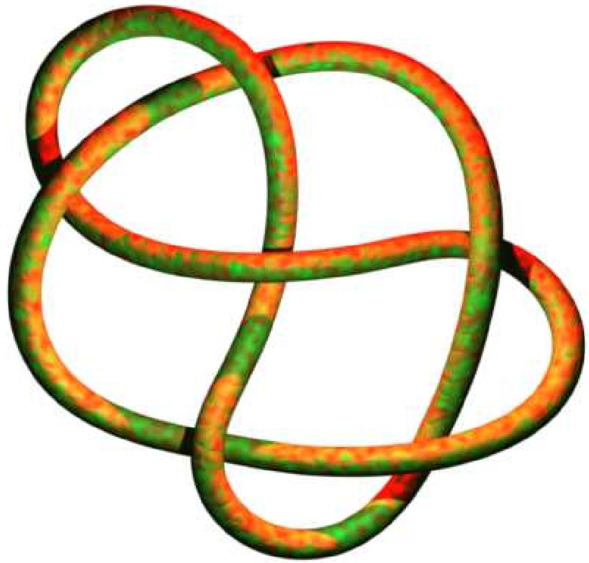

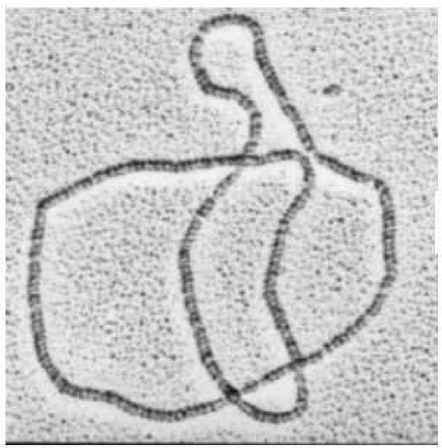

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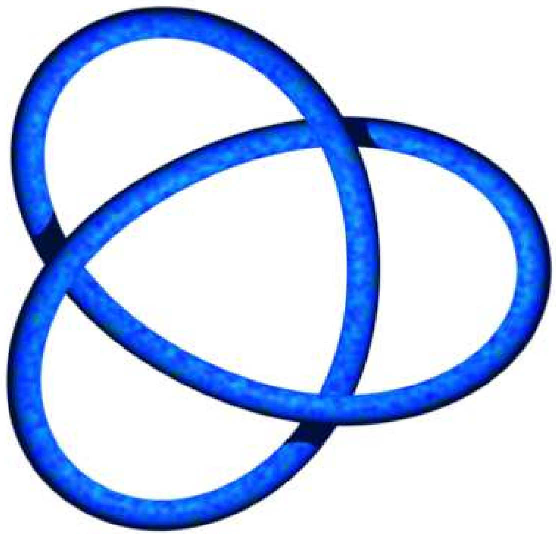
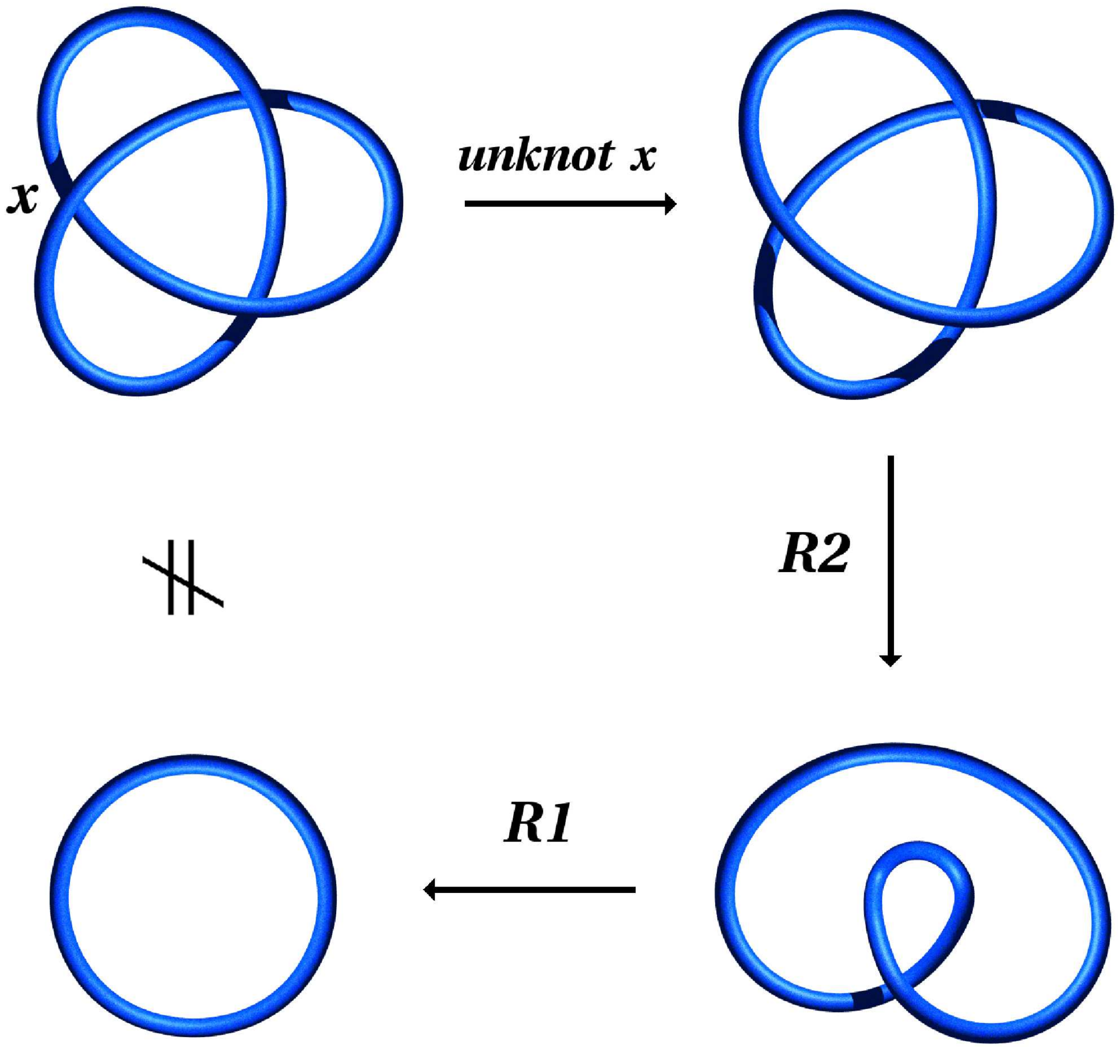
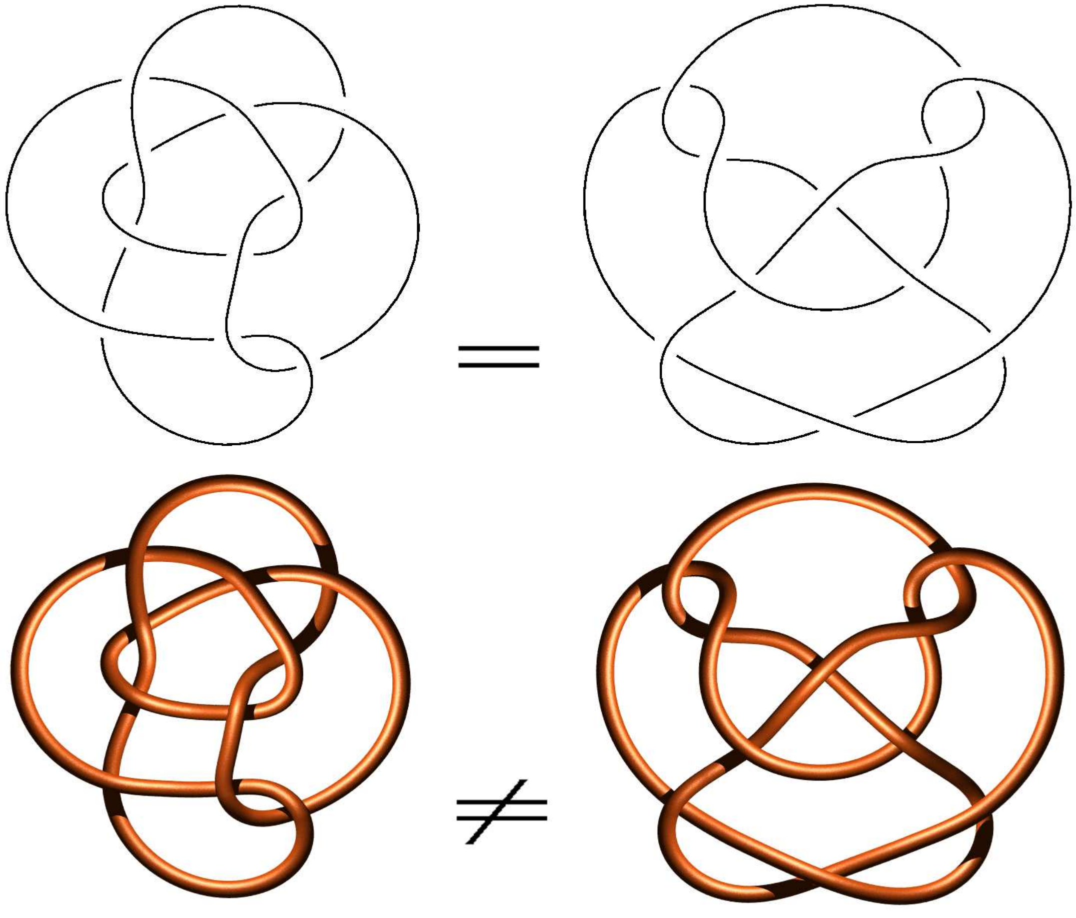


Unknotting Surgery



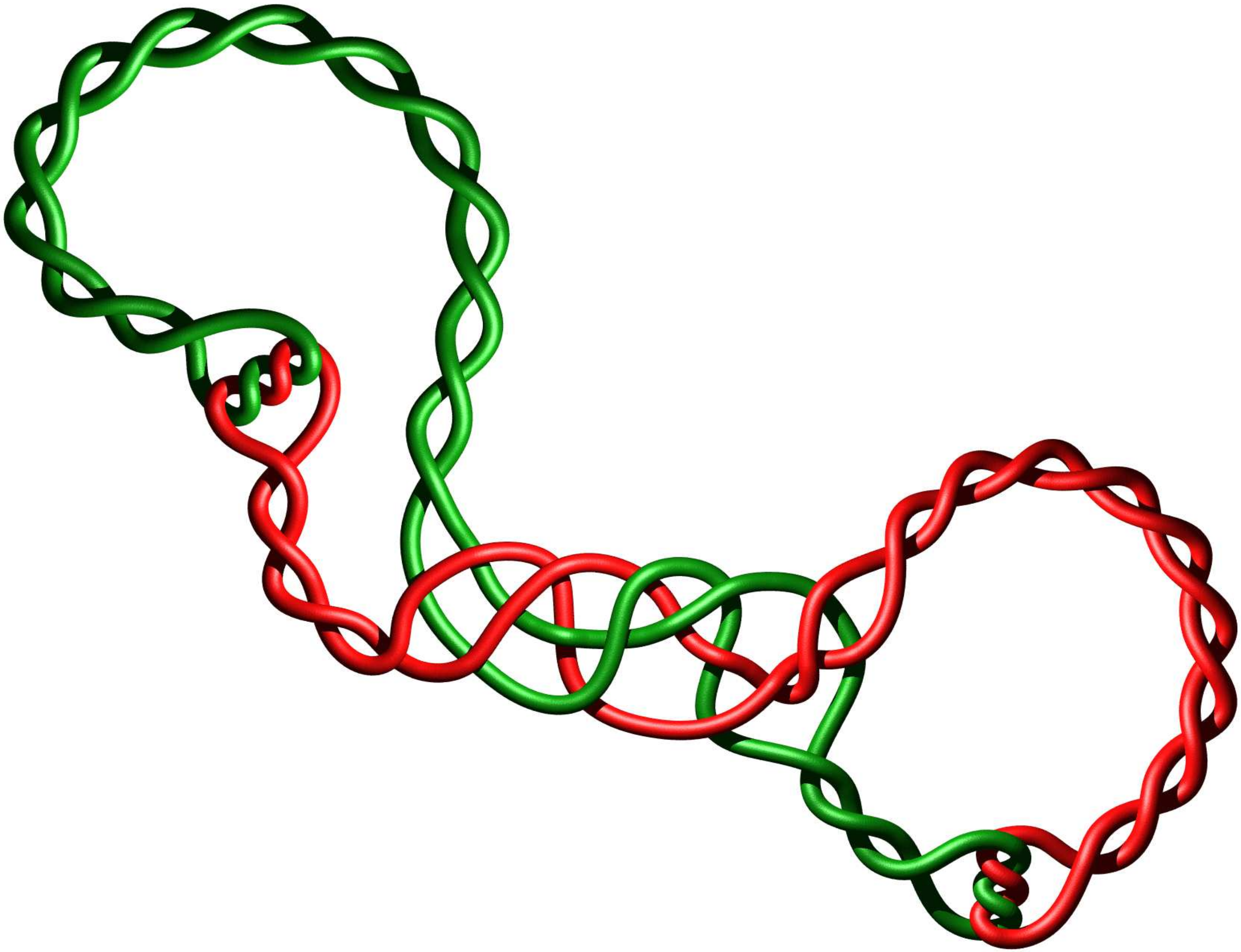
Physical Knots

- The body remembers.

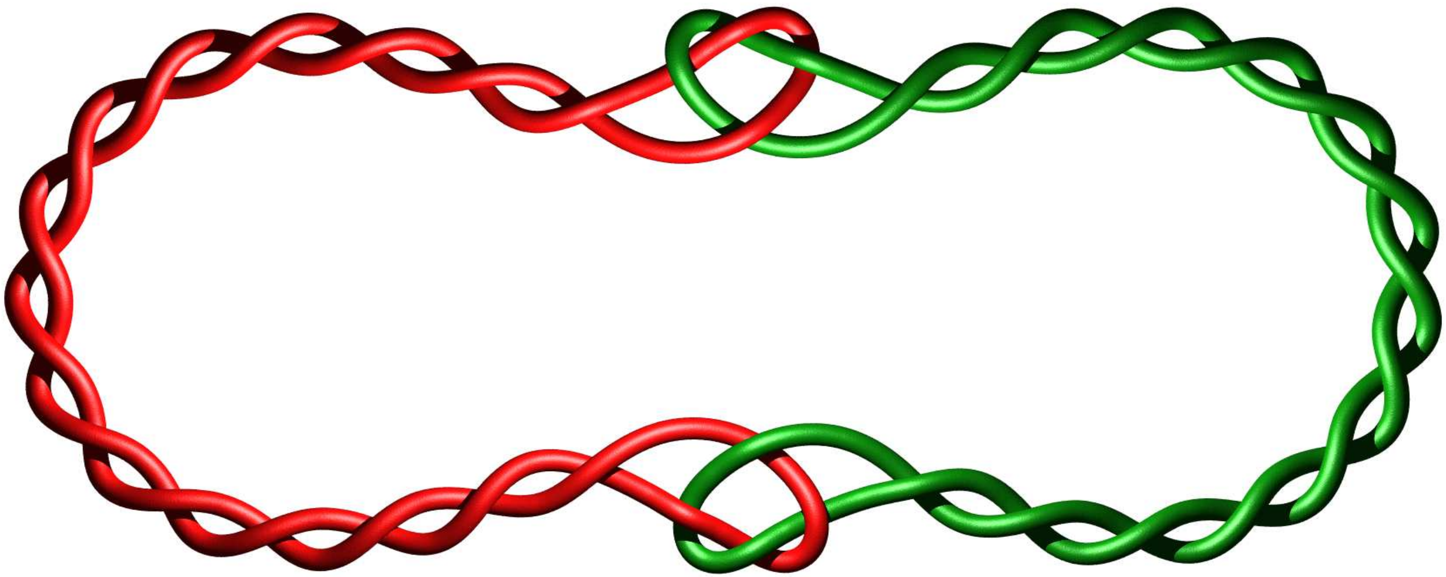


The famous Perko pair

Synthetic DNA

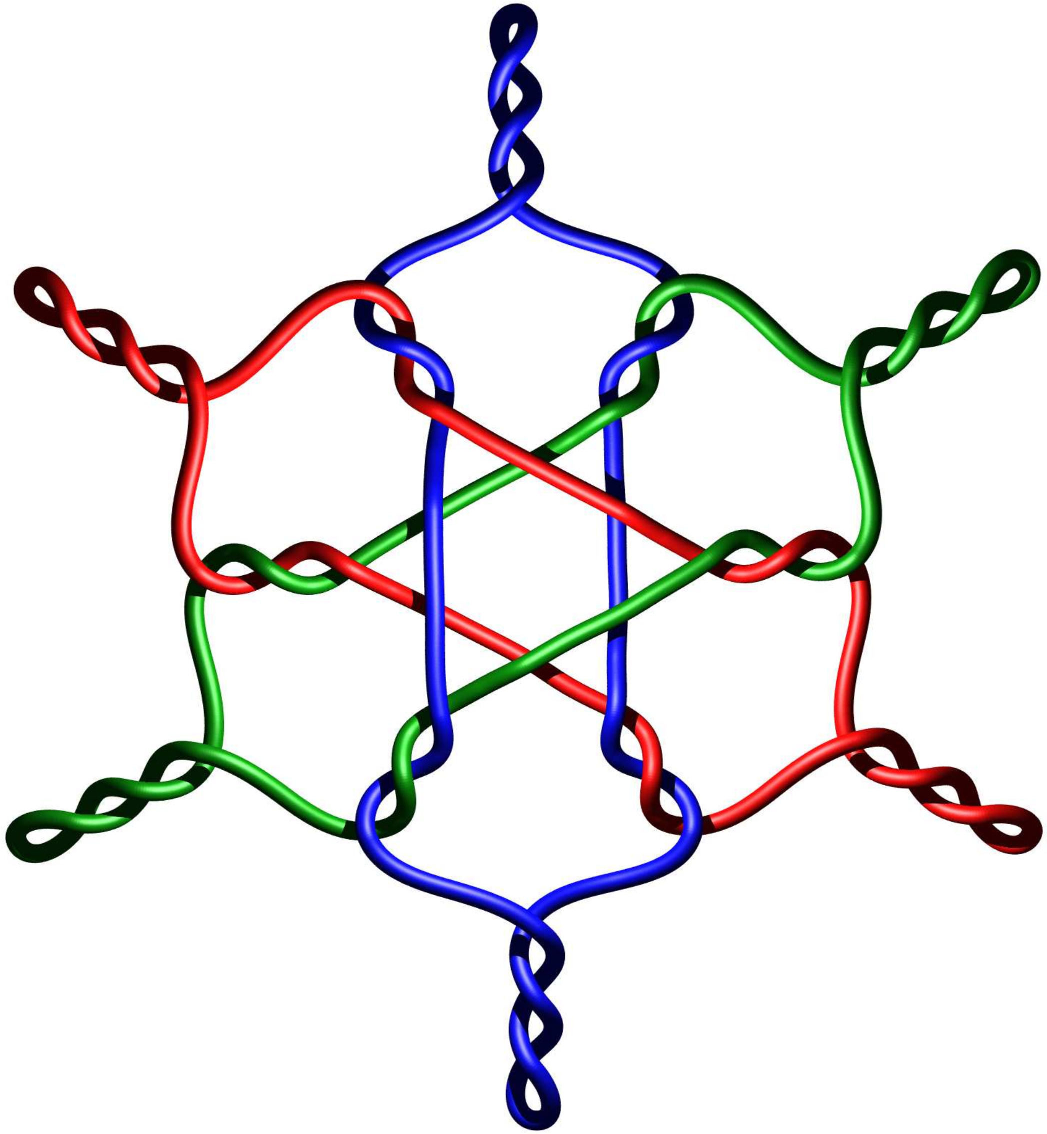


- A minimal (split group 14/13) configuration with 35 crossings.



$$1_1^0, -3_1^0, 1_1^1, -27_1^0, -1_2^0, 3_2^0, -1_2^1, -27_1^0 : 1_1^0, -3_1^0, 1_1^1, -27_1^1, -1_2^1, 3_2^0, -1_2^0, -27_1^1$$

- Nugatory stacks attract.



The Future of Engineering

- Benign structures with desirable properties.
- A nanotech revolution.
- Intelligent design (us) for immune system enhancement.

The End