The Density of Rational Points on Cayley's Cubic Surface

D.R. Heath-Brown
Mathematical Institute, Oxford

Abstract

The Cayley cubic surface is given by the equation $\sum_{i=1}^{4} X_i^{-1} = 0$. We show that the number of non-trivial rational points of height at most B is of exact order $B(\log B)^6$, as predicted by Manin's conjecture.

1 Introduction

The Cayley cubic surface is defined in $\mathbb{P}^3$ by the equation

$$\frac{1}{X_1}+\frac{1}{X_2}+\frac{1}{X_3}+\frac{1}{X_4}=0$$

or equivalently by

$$C:\ C(X_1,X_2,X_3,X_4)=X_2X_3X_4+X_1X_3X_4+X_1X_2X_4+X_1X_2X_3=0.$$

It has four singularities at the points $(1,0,0,0),(0,1,0,0),(0,0,1,0)$ and $(0,0,0,1)$. Moreover there are exactly 9 lines in the surface, and all of these are defined over the rationals. Three of the lines have the form $X_i+X_j=X_k+X_l=0$, while the remaining six have the shape $X_i=X_j=0$. We shall write U for the complement of these lines in the surface C.

The aim of this paper is to consider the density of rational points on the surface C. It transpires that "most" of the rational points lie on one of the lines described above. We shall think of such points as being "trivial" and exclude them from our counting function. We therefore define

$$N^*(B)=\#\{\mathbf{x}\in\mathbb{Z}^4:\ \mathbf{x}\in U,\ \max|x_i|\le B\},$$

where $\mathbf{x}=(x_1,x_2,x_3,x_4)$. Indeed, since vectors $\mathbf{x}$ which are scalar multiples of each other represent the same projective point, it is natural to consider only primitive vectors $\mathbf{x}$. (A vector $\mathbf{x}$ is said to be primitive if h.c.f.$(x_1, x_2, x_3, x_4) = 1$.) With this in mind we set

$$N(B) = \#\{\mathbf{x} \in \mathbb{Z}^4 : \ \mathbf{x} \in U, \ \max|x_i| \leq B, \ \mathbf{x} \text{ primitive}\}.$$

The corresponding number of rational points in $\mathbb{P}^3$ of "height" at most B is $\frac{1}{2}N(B)$, since $\mathbf{x}$ and $-\mathbf{x}$ represent the same point. Our two counting functions are closely related, since

$$N^*(B) = \sum_{h \leq B} N(B/h),$$

as one readily verifies.

Manin (see [7] and [1]) has given a very general conjecture which would predict in our case that

$$N(B) \sim cB(\log B)^6,$$

for a suitable positive constant c. Indeed, for surfaces with mild ("canonical") singularities (rational double points), one expects something of this type, with the the exponent of the logarithm being one less than the rank of the Picard group of a minimal resolution, which in our case is 7. Even though there is ample numerical evidence for the validity of this conjecture for cubic surfaces (see Heath-Brown [11] and Peyre and Tschinkel [15]), it has only been established for a small number of extremely simple cases, namely, forms of the toric cubic surface $X_1X_2X_3 = X_4^3$, see Batyrev and Tschinkel [2], de la Bretèche [3], Fouvry [6], Heath-Brown and Moroz [13] and Salberger [16]. The Cayley surface, while still singular, is considerably more intricate than any previous example. The goal of the present paper is to establish the following estimates.

Theorem *We have*

$$B(\log B)^6 \ll N(B) \ll B(\log B)^6.$$

Of the two inequalities here, the lower bound is relatively easy to prove. Indeed Slater and Swinnerton-Dyer [17] have established the lower bound corresponding to Manin's conjecture for any non-singular cubic surface defined over $\mathbb{Q}$, providing that it contains two skew lines defined over $\mathbb{Q}$. Although our surface is singular, it does contain several pairs of skew lines, and these are crucial to our argument. It would have been somewhat easier to have established upper bounds of order $B^{1+\varepsilon}$, with an arbitrary positive constant ε, or indeed of order $B(\log B)^A$ for some large constant A. However to achieve the correct exponent 6 requires more work.

It is natural to ask how close we come to establishing an asymptotic formula for $N(B)$. An analysis of the argument in §6 shows that the difficulty arises through our use of Lemma 6, which gives an upper bound for the number of primitive lattice points in $\mathbb{Z}^3$, lying in a box, and which satisfy a given linear equation. It is not obvious how one could formulate a useful version of this which replaced the upper bound by an asymptotic formula.

This paper uses a wide range of techniques. The two most important are the passage to the universal torsor, and the application of lattice methods to count solutions of ternary linear equations efficiently. Universal torsors were introduced by Colliot-Thélène and Sansuc [4], [5], and their use in counting points of bounded height is due to Peyre [14] and Salberger [16], amongst others. In our case the necessary result is given essentially by Lemma 1. The minimal desingularization of the Cayley cubic is the variety $V \subset \mathbb{P}^3 \times \mathbb{P}^3$ given by the equations

$$v_1x_1 = v_2x_2 = v_3x_3 = v_4x_4 = 0$$

$$v_1 + v_2 + v_3 + v_4 = 0,$$

and the variables y_i, z_{ij} and v_{ij} introduced at the end of §2 correspond to integral points on the (unique) universal torsor of V. Moreover the height which we use then corresponds to the anticanonical height on V. Since the rank of the Picard group of V is indeed 7, the exponent in our theorem does indeed correspond to the prediction by Manin and Batyrev.

The problem of counting solutions of ternary linear equations occurs a number of times. Our most fundamental result on this is Lemma 6, for which see the author's work [9]. The proof uses simple considerations about lattices and their successive minima.

Various other tools are used in a subsidiary capacity in the proof of the theorem. While it is not clear that these are essential to the argument, they demonstrate the diversity of methods that can be brought into play in this type of analysis. Thus we employ simple sieve ideas in §3, to handle coprimality conditions; Deligne's bounds for multidimensional exponential sums, via their application in the proof of Lemma 5 in [8] and [10]; and the author's large sieve estimate for real character sums [12], in the proof of Lemma 4.

Notation. One point of notation should be clarified. We shall use $\mathbb{N}$ to denote the set of *positive* integers. Some readers will be used to a notation in which $0 \in \mathbb{N}$, but we have found it convenient to adopt the alternative convention.

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2 The Universal Torsor

Our goal in this section is to use factorization information to analyze the equation $C(\mathbf{x}) = 0$, introducing further variables which will be of smaller size than the original variables $x_1, \ldots, x_4$, and which will satisfy additional equations. Although we shall not make any use of the fact, we note that these new variables describe the 'Universal Torsor' for the Cayley cubic. For the purposes of this analysis it will be convenient to introduce the convention that the letters i, j, k, l will denote generic distinct indices from the set $\{1, 2, 3, 4\}$.

It is useful to begin by observing that none of the variables x_i can vanish. For if $x_i = 0$, then the equation $C(\mathbf{x}) = 0$ implies that $x_j x_k x_l = 0$, so that the point $\mathbf{x}$ must lie on one of the excluded lines $X_i = X_j = 0$. We now set

$$y_i = \text{h.c.f.}(x_j, x_k, x_l), \tag{2.1}$$

The requirement that $\mathbf{x}$ is primitive is then equivalent to the condition

$$\text{h.c.f.}(y_i, y_j) = 1, \tag{2.2}$$

According to our convention this should be taken to mean that y_i and y_j are coprime whenever i and j are distinct. Since y_j, y_k, y_l are pairwise coprime and all divide x_i, their product divides x_i, and similarly for the other indices. We may therefore set

$$x_i = y_j y_k y_l z_i.$$

The definition (2.1) now reduces to

$$\text{h.c.f.}(y_k y_l z_j, y_j y_l z_k, y_j y_k z_l) = 1,$$

In view of (2.2), this is equivalent to the two conditions

$$\text{h.c.f.}(y_i, z_i) = 1, \tag{2.3}$$

and

$$\text{h.c.f.}(z_i, z_j, z_k) = 1. \tag{2.4}$$

Moreover the equation $C(\mathbf{x}) = 0$ becomes

$$z_2z_3z_4y_1 + z_1z_3z_4y_2 + z_1z_2z_4y_3 + z_1z_2z_3y_4 = 0 \tag{2.5}$$

on recalling that none of $x_1, \ldots, x_4$ can vanish.

Our problem is therefore reduced to counting solutions of the equation (2.5), lying in the region

$$|y_jy_ky_lz_i| \leq B,$$

and subject to the constraints (2.2), (2.3) and (2.4). Moreover solutions in which any of the variables is zero are to be discounted, since they produce points $\mathbf{x}$ on one of the lines in the surface C. Similarly solutions with

$$z_jz_kz_ly_i + z_iz_kz_ly_j = 0$$

are to be discounted.

We now perform a second reduction. We begin by defining

$$z_{ij} = z_{ji} = \text{h.c.f.}(z_i, z_j). \tag{2.6}$$

In view of (2.4) we have

$$\text{h.c.f.}(z_{ij}, z_{ik}) = 1.$$

Since z_{ij}, z_{ik}, z_{il} all divide z_i, and are coprime in pairs, it follows that their product divides z_i. We may therefore write

$$z_i = B_iw_i$$

where

$$B_i = z_{ij}z_{ik}z_{il}. \tag{2.7}$$

The definition (2.6) then reduces to

$$\text{h.c.f.}(z_{ik}z_{il}w_i, z_{jk}z_{jl}w_j) = 1,$$

or equivalently

$$\text{h.c.f.}(w_i, w_j) = 1, \tag{2.8}$$

$$\text{h.c.f.}(w_i, z_{jk}) = 1, \tag{2.9}$$

and

$$\text{h.c.f.}(z_{ab}, z_{cd}) = 1, \quad (\{a, b\}, \{c, d\} \text{ distinct}). \tag{2.10}$$

The equation (2.5) now becomes

$$A_1w_2w_3w_3y_1 + A_2w_1w_3w_4y_2 + A_3w_1w_2w_4y_3 + A_4w_1w_2w_3y_4 = 0, \qquad (2.11)$$

where

$$A_i = z_{jk}z_{jl}z_{kl}. \qquad (2.12)$$

We therefore see that $w_i | A_i w_j w_k w_l y_i$. In view of (2.8) and (2.9) this implies that $w_i | y_i$. Since $w_i | z_i$ we conclude from (2.3) that $w_i = \pm 1$.

We now have

$$x_i = B_i y_j y_k y_l w_i,$$

with $w_i = \pm 1$. However, in making the definitions (2.1) and (2.6), the highest common factors are only defined up to sign. Let us assume, temporarily, that we chose the variables y_i and z_{ij} to be positive. We proceed to replace each y_i by $w_i y_i$, whence

$$x_i = \varepsilon B_i y_j y_k y_l,$$

with $\varepsilon = w_1w_2w_3w_4 = \pm 1$. Thus, if we replace $\mathbf{x}$ by $\varepsilon\mathbf{x}$, we obtain

$$x_i = B_i y_j y_k y_l, \qquad (2.13)$$

where the variables z_{ij} are positive but y_i may be of either sign. After these changes the equation (2.11) reduces to

$$A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4 = 0. \qquad (2.14)$$

Moreover the condition (2.4) is implied by (2.10), while (2.3) is equivalent to

$$\text{h.c.f.}(y_i, z_{ij}) = 1. \qquad (2.15)$$

We may therefore summarize our conclusions as follows.

Lemma 1 *Let $\mathbf{x} \in U$ be a primitive integral solution of $C(\mathbf{x}) = 0$. Then either $\mathbf{x}$ or $-\mathbf{x}$ takes the form (2.13), with non-zero integer variables y_i and positive integer variables z_{ij} constrained by the conditions (2.2), (2.10) and (2.15), and satisfying the equation (2.14). Moreover none of $A_1y_1 + A_2y_2$, $A_1y_1 + A_3y_3$ or $A_1y_1 + A_4y_4$ may vanish.*

Conversely, if y_i and z_{ij} are as above, then the vector $\mathbf{x}$ given by (2.13) will be a primitive integral solution of $C(\mathbf{x}) = 0$ lying in U.

To proceed further, we note that the equation (2.14) implies that

$$z_{ij} | z_{kl}(z_{ik}z_{il}y_j + z_{jk}z_{jl}y_i),$$

whence (2.10) yields

$$z_{ij} | z_{ik} z_{il} y_j + z_{jk} z_{jl} y_i.$$

We therefore write

$$z_{ik} z_{il} y_j + z_{jk} z_{jl} y_i = z_{ij} v_{ij}, \tag{2.16}$$

so that equation (2.14) is equivalent to each of the relations

$$v_{ij} + v_{kl} = 0.$$

Note that $v_{ij} = v_{ji}$, since z_{ij} is also symmetric in the indices ij. We now calculate that

$$\begin{aligned} z_{ij} v_{ij} z_{ik} v_{ik} &= (z_{ik} z_{il} y_j + z_{jk} z_{jl} y_i)(z_{ij} z_{il} y_k + z_{jk} z_{kl} y_i) \\ &= z_{il}^2 z_{ij} z_{ik} y_j y_k + z_{jk} y_i (z_{ik} z_{il} z_{kl} y_j + z_{jl} z_{ij} z_{il} y_k + z_{jk} z_{jl} z_{kl} y_i), \end{aligned}$$

whence (2.14) yields

$$z_{ij} v_{ij} z_{ik} v_{ik} = z_{il}^2 z_{ij} z_{ik} y_j y_k - z_{jk} y_i . z_{ij} z_{ik} z_{jk} y_l.$$

We therefore conclude that

$$v_{ij} v_{ik} = z_{il}^2 y_j y_k - z_{jk}^2 y_i y_l. \tag{2.17}$$

3 The Lower Bound

To tackle the lower bound problem in our theorem we begin by considering solutions for which the variables z_{ij} are fixed, and relatively small, while the variables y_i are comparatively large, and lie in the dyadic ranges

$$Y_i < |y_i| \leq 2Y_i. \tag{3.1}$$

In the notation given by (2.7) and (2.12) we observe that the condition $\max |x_i| \leq B$ is equivalent to

$$A_i A_j A_k |y_i y_j y_k| \leq BP, \tag{3.2}$$

where

$$P = z_{12} z_{13} z_{14} z_{23} z_{24} z_{34} = A_i B_i. \tag{3.3}$$

We shall choose

$$Y_i = [\frac{(BP)^{1/3}}{2A_i}], \tag{3.4}$$

We will then have (3.2) whenever the y_i lie in the ranges (3.1). We shall assume moreover that

$$P \leq B^{\delta}. \tag{3.5}$$

where δ is a small positive constant to be specified later, see (3.14). We shall write

$$\mathcal{N} = \mathcal{N}(z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34})$$

for the number of solutions (y_1, y_2, y_3, y_4) of (2.14), in the ranges (3.1), subject to the constraints (2.2) and (2.15), and not on any of the lines $A_1y_i + A_iy_i = 0$.

The main difficulty in establishing our lower bound comes from the coprimality conditions (2.2) and (2.15). To handle these we begin by setting

$$Q = P \prod_{p \leq \sqrt{\log B}} p$$

and writing $\mathcal{N}_1$ for the number of solutions in which (2.2) is replaced by the weaker condition

$$(y_i, y_j, Q) = 1, \quad \text{for all } \ i \neq j.$$

We take $\mathcal{N}_2$ to be the number of solutions in which some pair y_i, y_j has a prime factor $p|y_i, y_j$ with $p \nmid Q$. Clearly we then have

$$\mathcal{N} \geq \mathcal{N}_1 - \mathcal{N}_2. \tag{3.6}$$

We begin by estimating $\mathcal{N}_1$, and first note that there can be at most

$$Y_1 Y_j \ll (BP)^{2/3} \ll B^{2/3 + 2\delta/3} \tag{3.7}$$

solutions on one of the lines $A_1y_1 + A_iy_i = A_jy_j + A_ky_k = 0$, by (3.4) and (3.5). This bound will turn out to be of negligible size. We can therefore ignore the condition that solutions may not lie on such a line. We now proceed by picking out the coprimality conditions with the Möbius function. Let

$$\mathcal{N}_3 = \mathcal{N}_3(d_1, \ldots, d_4; d_{12}, \ldots, d_{34})$$

denote the number of solutions of the equation (2.14), with y_i in the ranges (3.1), and such that $d_i|y_i$ and $d_{ij}|y_i, y_j$ for every choice of indices. Then

$$\mathcal{N}_1 = \sum_{d_i | B_i} \mu(d_1) \ldots \mu(d_4) \sum_{d_{ij} | Q} \mu(d_{12}) \ldots \mu(d_{34}) \mathcal{N}_3, \tag{3.8}$$

since the condition (2.15) is equivalent to h.c.f.$(y_i, B_i) = 1$. We must now estimate $\mathcal{N}_3$. We shall write h_1 for the lowest common multiple of d_{12}, d_{13}, d_{14} and d_1, and similarly for h_2, h_3 and h_4. We may then re-interpret $\mathcal{N}_3$ as the number of integer triples (n_1,n_2,n_3) for which $A_ih_i|n_i$ and $A_4h_4|n_1+n_2+n_3$, and which lie in the region

$$\mathcal{R}:\ A_iY_i<|n_i|\le 2A_iY_i\quad (i\le 3),\quad A_4Y_4<|n_1+n_2+n_3|\le 2A_4Y_4. \quad (3.9)$$

The divisibility conditions define an integer sublattice $\Lambda\le\mathbb{Z}^3$, such that

$$\mathcal{N}_3=\#(\Lambda\cap\mathcal{R}).$$

We shall need to compute the determinant of Λ, or, what is the same thing, its index in $\mathbb{Z}^3$. This is most easily done locally. Write

$$A_ih_i=\prod_p p^{\nu(p,i)}$$

and let $\Lambda_p\le\mathbb{Z}^3$ be the lattice for which $p^{\nu(p,i)}|n_i$ and $p^{\nu(p,4)}|n_1+n_2+n_3$. Let $\nu(p,0)=\min_i\nu(p,i)$ and let $\Lambda_p^{(0)}\le\mathbb{Z}^3$ be defined by the conditions $p^{\nu(p,i)-\nu(p,0)}|n_i$ and $p^{\nu(p,4)-\nu(p,0)}|n_1+n_2+n_3$. Then a moment's thought reveals that

$$\det(\Lambda_p^{(0)})=p^{\{\nu(p,1)-\nu(p,0)\}+\{\nu(p,2)-\nu(p,0)\}+\{\nu(p,3)-\nu(p,0)\}+\{\nu(p,4)-\nu(p,0)\}},$$

and

$$\det(\Lambda_p)=p^{3\nu(p,0)}\det(\Lambda_p^{(0)}),$$

whence

$$\det(\Lambda_p)=\frac{\prod_i p^{\nu(p,i)}}{\text{h.c.f.}(p^{\nu(p,i)})}.$$

We now observe that Λ is the intersection of the various Λ_p, which have pairwise coprime indices in $\mathbb{Z}^3$. It therefore follows that

$$\det(\Lambda)=\prod_p\frac{\prod_i p^{\nu(p,i)}}{\text{h.c.f.}(p^{\nu(p,i)})}=\frac{\prod_i A_ih_i}{\text{h.c.f.}(A_ih_i)}.$$

We may choose a basis $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3$ of Λ with $|\mathbf{b}_i|\ll\det(\Lambda)$. Taking M to be the 3×3 integer matrix formed from the vectors $\mathbf{b}_i$ we see that $\Lambda=M\mathbb{Z}^3$, and that $\det(M)=\det(\Lambda)$. If $\mathcal{R}$ is the region (3.9) then

$$\#(\Lambda\cap\mathcal{R})=\#(\mathbb{Z}^3\cap M^{-1}\mathcal{R}).$$

However $M^{-1}\mathcal{R}$ has volume $\text{meas}(\mathcal{R})/\det(M)$, is bounded by $O(1)$ planar sides, and lies in a sphere of radius r, say, where $r\ll||M^{-1}||\max Y_i$. Here

$||M^{-1}||$ is the modulus of the largest entry in M^{-1}, so that $||M^{-1}|| \ll \det(\Lambda)$. It follows that

$$\mathcal{N}_3 = \#(\Lambda \cap \mathcal{R}) = \#(\mathbb{Z}^3 \cap M^{-1}\mathcal{R}) = \frac{\mathrm{meas}(\mathcal{R})}{\det(M)} + O(r^2).$$

Since $d_i|B_i$ and $d_{ij}|Q$ we have $A_ih_i \le A_iB_iQ = PQ$, by (3.3). Thus (3.5) yields $\det(\Lambda) \le P^4Q^4 \ll B^{12\delta}$, since

$$Q \ll P\exp(O(\sqrt{\log B})) \ll B^{2\delta}.$$

We therefore deduce that

$$\mathcal{N}_3 = \mathrm{meas}(\mathcal{R})\frac{\mathrm{h.c.f.}(A_ih_i)}{\prod_i A_ih_i} + O(B^{2/3+25\delta}).$$

We now insert this into (3.8), so that

$$\begin{aligned}\mathcal{N}_1 &= \mathrm{meas}(\mathcal{R})\sum_{d_i,d_{ij}} \mu(d_1)\ldots\mu(d_4)\mu(d_{12})\ldots\mu(d_{34})\frac{\mathrm{h.c.f.}(A_ih_i)}{\prod_i A_ih_i} \\ &\qquad + O(B^{2/3+26\delta}), \end{aligned} \tag{3.10}$$

since the usual estimate for the divisor function shows that there are $O(B^\delta)$ divisors d_i, d_{ij} in total.

It remains to consider the sum

$$\sum_{d_i,d_{ij}} \mu(d_1)\ldots\mu(d_4)\mu(d_{12})\ldots\mu(d_{34})\frac{\mathrm{h.c.f.}(A_ih_i)}{\prod_i A_ih_i}.$$

By multiplicativity we see that this is a product of local factors e_p, say. For primes $p \nmid Q$ we clearly have $e_p = 1$. When $p|Q$ but $p \nmid P$ we define the integer N, temporarily, as the number of quadruples $(x_1, x_2, x_3, x_4) \pmod p$ satisfying $\mathrm{h.c.f.}(x_i, x_j, p) = 1$ for $i \ne j$, and such that $x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod p$. We then find, again using the Möbius function, that $N = p^3 e_p$. An easy computation then yields

$$e_p = 1 - \frac{6}{p^2} + \frac{5}{p^3}. \tag{3.11}$$

For the remaining primes p, namely those for which $p|P$, we note that p will divide exactly one z_{ij}, by (2.10), and we suppose without loss of generality that z_{12} contains p with exponent $e \ge 1$, say. We then let $A'_i = A_i$ if $i = 1$ or 2, and $A'_i = p^{1-e}A_i$ for $i = 3$ or 4. Since $p^2 \nmid A_ih_i$ for $i = 1$ or 2 we have

$$\nu_p(\mathrm{h.c.f.}(A_ih_i)) = \nu_p(\mathrm{h.c.f.}(A'_ih_i)),$$

where ν_p is the p-adic valuation, and hence

$$\nu_p\left(\frac{\text{h.c.f.}(A_ih_i)}{\prod_i A_ih_i}\right) = \nu_p\left(p^{2-2e}\frac{\text{h.c.f.}(A_i'h_i)}{\prod_i A_i'h_i}\right).$$

Now let N denote, temporarily, the number of quadruples $(x_1,x_2,x_3,x_4)\in\mathbb{N}^4$ satisfying the conditions

$$x_1,x_2\le p^2,\quad x_3,x_4\le p,$$

$$\text{h.c.f.}(x_i,x_j,p)=1 \text{ for } i\ne j,\quad \text{h.c.f.}(x_1,p)=\text{h.c.f.}(x_2,p)=1,$$

and

$$x_1+x_2+px_3+px_4\equiv 0 \pmod{p^2}.$$

We then find, using the Möbius function once more, that $N=p^{2e+4}e_p$, and another easy computation then produces

$$e_p=(1-\frac{1}{p})(1-\frac{1}{p^2})p^{-2e}. \tag{3.12}$$

The formulae (3.11) and (3.12) show that

$$\sum_{d_i,d_{ij}}\mu(d_1)\dots\mu(d_4)\mu(d_{12})\dots\mu(d_{34})\frac{\text{h.c.f.}(A_ih_i)}{\prod_i A_ih_i}\gg P^{-2}\frac{\phi(P)}{P},$$

and since we clearly have $\text{meas}(\mathcal{R})\gg BP$, from (3.4) and (3.9), we deduce from (3.5), (3.7) and (3.10) that

$$\mathcal{N}_1\gg\frac{B}{P}\frac{\phi(P)}{P} \tag{3.13}$$

providing that we take

$$\delta=\frac{1}{84}. \tag{3.14}$$

We turn now to $\mathcal{N}_2$, which we must estimate from above. We start by considering the contribution from solutions in which $p|y_1,y_2$, say, with $p\nmid Q$ and $R<p\le 2R$. We begin with the following preliminary observations. Clearly there are no solutions with $R\gg Y_1$, and so we may suppose that $R\ll Y_1$. Moreover, if $\delta\le 1/7$ then (3.3), (3.4) and (3.5) yield

$$z_{12}\le\frac{P}{A_1}\le\frac{(BP)^{1/6}}{A_1}\ll Y_1^{1/2}.$$

It therefore follows that

$$1 \ll (\frac{Y_1}{R})^{1/2} \frac{Y_1^{1/2}}{z_{12}} = \frac{Y_1}{R^{1/2} z_{12}}.$$

Thus

$$1 + \frac{Y_1}{R z_{12}} \ll \frac{Y_1}{R^{1/2} z_{12}}. \tag{3.15}$$

Since we are seeking an upper bound for $\mathcal{N}_2$, the coprimality conditions can be dropped. If we set $y_1 = pt_1, y_2 = pt_2$ then we have $Y_i/R \ll |t_i| \ll Y_i/R$ for $i = 1, 2$. Moreover, since $z_{12}|A_3, A_4$ we have $A_1 t_1 \equiv -A_2 t_2 \pmod{z_{12}}$. However z_{12} is coprime to A_1, so that each admissible value of t_2 determines $O(1 + Y_1/(R z_{12}))$ values of t_1. It therefore follows from (3.15) that there are $O(Y_1 Y_2 R^{-3/2} z_{12}^{-1})$ possible pairs t_1, t_2.

For each such pair we now estimate how many triples p, y_3, y_4 there might be. We put $A_1 t_1 + A_2 t_2 = z_{12} s$. Then

$$ps + \frac{A_3}{z_{12}} y_3 + \frac{A_4}{z_{12}} y_4 = 0,$$

whence

$$ps + \frac{A_3}{z_{12}} y_3 \equiv 0 \ (\mathrm{mod}\ \frac{A_4}{z_{12}}).$$

Since A_3/z_{12} and A_4/z_{12} are coprime, by (2.10), we see that each value of p determines y_3 modulo A_4/z_{12}, producing $O(1 + Y_3 z_{12}/A_4)$ values. However if $\delta \leq 1/7$ then (3.4) and (3.5) suffice to show that $1 \ll Y_3/A_4$, so we will have $O(R Y_3 z_{12}/A_4)$ possible pairs p, y_3, each of which determines at most one admissible y_4.

These bounds show that the range $R < p \leq 2R$ contributes

$$\frac{Y_1 Y_2}{R^{3/2} z_{12}} \frac{R Y_3 z_{12}}{A_4} = \frac{A_1 Y_1 . A_2 Y_2 . A_3 Y_3}{P^2 R^{1/2}} \ll \frac{B}{P R^{1/2}}$$

to $\mathcal{N}_2$. If we now sum $R \gg \sqrt{\log B}$ over powers of two we deduce that

$$\mathcal{N}_2 \ll \frac{B}{P} (\log B)^{-1/4}.$$

Since $\phi(P)/P \gg (\log\log P)^{-1}$ we deduce from (3.13) that $\mathcal{N}_2 = o(\mathcal{N}_1)$ and hence, via (3.6), that

$$\mathcal{N} \gg \frac{B}{P} \frac{\phi(P)}{P}.$$

We summarize our conclusions thus far as follows.

Lemma 2 *For a given admissible set of values* $z_{12}, \ldots, z_{34}$ *satisfying*

$$P = \prod z_{ij} \leq B^{1/84}$$

there are

$$\gg \frac{B}{P} \frac{\phi(P)}{P}$$

corresponding values of y_1, y_2, y_3, y_4*.*

To complete the proof of the lower bound part of our theorem, we observe that any square-free value of P will factorize into values $z_{12}, \ldots, z_{34}$ satisfying (2.10) and (3.3) in exactly $d_6(P)$ ways. (Here $d_6(\ldots)$ is the generalized divisor function.) Thus

$$N(B) \gg \sum_{P \leq B^{1/84}} \mu(P)^2 d_6(P) \frac{B}{P} \frac{\phi(P)}{P}$$

and a standard estimation using Perron's formula then produces the required bound

$$N(B) \gg B(\log B)^6.$$

4 The Upper Bound—Basic Estimates

In contrast to the work of the previous section, in giving an upper bound for $N(B)$ we can ignore questions of coprimality whenever we wish to do so. Instead our principal technical problem will be to control precisely the number of logarithms appearing in our estimates.

We shall need to understand the equations (2.16) and (2.17), and our results are summarized as follows.

Lemma 3 *Let real numbers* $K_1, \ldots, K_7 > 0$ *be given, and let* N_1 *denote the number of solutions* $n_i \in \mathbb{N}$ *to the equation*

$$n_1 n_2 n_3 + n_4 n_5 n_6 = n_7 n_8 \quad (K_i < n_i \leq 2K_i, \quad 1 \leq i \leq 7)$$

subject to the condition

$$\text{h.c.f.}(n_1 n_2 n_3\, ,\ n_4 n_5 n_6) = 1. \tag{4.1}$$

Then

$$N_1 \ll K_1 K_2 K_3 K_4 K_5 K_6. \tag{4.2}$$

Similarly, if N_2 is the number of solutions of

$$n_1n_2n_3 = n_4n_5n_6 + n_7n_8 \tag{4.3}$$

under the same conditions, then

$$N_2 \ll K_1K_2K_3K_4K_5K_6.$$

Lemma 4 *Let real numbers $K_1, \ldots, K_7 > 0$ be given, and let N_3 denote the number of solutions $n_i \in \mathbb{N}$ of the equation*

$$n_1^2n_2n_3 + n_4^2n_5n_6 = n_7n_8 \quad (K_i < n_i \le 2K_i, \;\; 1 \le i \le 7) \tag{4.4}$$

subject to the condition (4.1). Then

$$N_3 \ll K_1K_2K_3K_4K_5K_6 \max\{(\frac{K_1^2K_2K_3}{K_4^2K_5K_6})^{1/4}, (\frac{K_4^2K_5K_6}{K_1^2K_2K_3})^{1/4}\}. \tag{4.5}$$

If N_4 is the corresponding number of solutions for the equation

$$n_1^2n_2n_3 - n_4^2n_5n_6 = n_7n_8 \quad (K_i < n_i \le 2K_i, \;\; 1 \le i \le 7) \tag{4.6}$$

we have

$$\begin{aligned} N_4 \;\; &\ll \;\; \{1 + \frac{\log K_1K_4}{(K_2K_3K_5K_6)^{1/3}}\}K_1K_2K_3K_4K_5K_6 \\ &\qquad\qquad \times \max\{(\frac{K_1^2K_2K_3}{K_4^2K_5K_6})^{1/4}, (\frac{K_4^2K_5K_6}{K_1^2K_2K_3})^{1/4}\}. \end{aligned} \tag{4.7}$$

We may think of the bound for N_1, for example, as describing the number of divisors of $n_1n_2n_3 + n_4n_5n_6$ which lie in specified dyadic ranges. Note that we do not impose a condition on the size of n_8. We may remark that in both lemmas we can use the standard bound for the divisor function to show that each 6-tuple $(n_1, \ldots, n_6)$ determines $O((\max K_i)^{\varepsilon})$ pairs of divisors n_7, n_8, for any fixed $\varepsilon > 0$. This immediately yields the bounds

$$N_1, N_2, N_3, N_4 \ll (K_1K_2K_3K_4K_5K_6)^{1+\varepsilon},$$

so that the important aspect of Lemma 3 is the removal of the exponent ε. It would be relatively easy to replace the ε power by a power of a logarithm, but this would be insufficient for our purposes. In relation to Lemma 4 we conjecture that the factor

$$\max\{(\frac{K_1^2K_2K_3}{K_4^2K_5K_6})^{1/4}, (\frac{K_4^2K_5K_6}{K_1^2K_2K_3})^{1/4}\}$$

may be removed in both cases. However it is not possible to delete the term

$$1+\frac{\log K_1K_4}{(K_2K_3K_5K_6)^{1/3}}$$

in our estimate for N_4. Indeed, when $K_1 = K_4 = K_7$ and $K_2 = K_3 = K_5 = K_6 = 1/2$ we easily find that $N_4 \gg K_1K_4\log(K_1K_4)$. Thus our bounds are not as sharp as we would like, but they are optimal in the critical case in which $K_1^2K_2K_3$ and $K_4^2K_5K_6$ have the same order of magnitude.

Before beginning the proofs of these results we observe that the condition (4.1) implies that the three terms $n_1n_2n_3$, $n_4n_5n_6$ and n_7n_8 are coprime in pairs. We shall use this fact repeatedly without further comment, in relation to both lemmas.

In this section we shall prove Lemma 3. The treatment of Lemma 4, which we defer to the next section, uses some of the same principles, but is much more involved. We begin by considering N_1. By the symmetry we may assume that

$$K_1K_2K_3 \gg K_4K_5K_6. \tag{4.8}$$

It is then clear that $N_1 = 0$ unless

$$\frac{K_1K_2K_3}{K_7} \ll n_8 \ll \frac{K_1K_2K_3}{K_7}, \tag{4.9}$$

as we shall now assume. We write this condition as $K_8 \ll n_8 \ll K_8$. We may then suppose, by symmetry, that $K_7 \geq K_8$, whence (4.9) implies that $K_8 \ll (K_1K_2K_3)^{1/2}$. We then apply the following estimate.

Lemma 5 *Let $K_1, K_2, K_3 > 0$ and let $q \ll (K_1K_2K_3)^{1/2}$. Then for any integer a coprime to q, we have*

$$\#\{(n_1,n_2,n_3)\in\mathbb{N}^3 :\ K_i < n_i \leq 2K_i,\ n_1n_2n_3 \equiv a \pmod{q}\}$$
$$\ll K_1K_2K_3/\phi(q).$$

We shall prove this in a moment. However if we apply it to the current situation we see, on taking $q = n_8$ and summing over n_4, n_5, n_6 and n_8, that

$$N_1 \ll K_1K_2K_3K_4K_5K_6 \sum_{K_8\ll n_8\ll K_8} \phi(n_8)^{-1}.$$

The required bound (4.2) now follows, since

$$\sum_{K_8\ll n_8\ll K_8} \phi(n_8)^{-1} \ll 1.$$

To handle N_2 we note that we automatically have (4.8) if there are to be any solutions. We can then proceed exactly as before providing that

$$K_1K_2K_3 \geq 16K_4K_5K_6,$$

since this is enough to ensure that (4.9) holds. It therefore remains to consider the case in which

$$K_1K_2K_3 \ll K_4K_5K_6 \ll K_1K_2K_3. \tag{4.10}$$

In this case we shall assume that

$$K_1 \geq K_2 \geq K_3, \quad \text{and} \quad K_4 \geq K_5 \geq K_6,$$

as we may, by the symmetry. It follows in particular that

$$K_2K_3K_5K_6 \leq (K_1K_2K_3K_4K_5K_6)^{2/3}. \tag{4.11}$$

We now write $N_{2,a}(q)$ for the the number of solutions $(n_1, \ldots, n_6)$ corresponding to each value $n_7 = q$, so that

$$N_2 \ll \sum_{K_7 < q \leq 2K_7} N_{2,a}(q). \tag{4.12}$$

Moreover, if we set

$$K_8 = K_1K_2K_3/K_7, \tag{4.13}$$

then it is apparent that we must have $n_8 \ll K_8$ in any solution of (4.3). Thus

$$N_2 \ll \sum_{q \ll K_8} N_{2,b}(q), \tag{4.14}$$

where $N_{2,b}(q)$ counts the solutions corresponding to a given value $n_8 = q$. We plan to use (4.12) when $K_7 \ll (K_1K_2K_3)^{1/2}$. If this condition fails to hold we must have $K_8 \ll (K_1K_2K_3)^{1/2}$, in which case we shall employ (4.14).

We now introduce the following result, which is part of Lemma 3 of the author's work [9].

Lemma 6 *Let $\mathbf{v} \in \mathbb{Z}^3$ be a primitive vector, and let $H_i > 0$ for $i = 1, 2, 3$ be given. Then the number of primitive vectors $\mathbf{x} \in \mathbb{Z}^3$ for which $\mathbf{v}.\mathbf{x} = 0$, and which lie in the box $|x_i| \leq H_i$ $(i = 1, 2, 3)$, is at most*

$$4 + 12\pi \frac{H_1H_2H_3}{\max H_i|v_i|} \leq 4 + 12\pi \frac{H_1H_2}{|v_3|}.$$

Recall that an integer vector is said to be primitive if its coordinates have no common factor. In our applications this condition will be a consequence of (2.2), (2.10) and (2.15).

To bound $N_{2,a}(q)$ we write the condition (4.3) as $\mathbf{v}.\mathbf{x} = 0$ where $\mathbf{v} = (n_2n_3, -n_5n_6, -q)$ and $\mathbf{x} = (n_1, n_4, n_8)$. We set $H_1 = 2K_1, H_2 = 2K_4$ and

$$H_3 = 8K_1K_2K_3/q.$$

Then Lemma 6 produces the bound $O(1+K_1K_4/q)$ for the number of triples (n_1, n_4, n_8) and it follows on summing over n_2, n_3, n_5, n_6 and q that

$$N_2 \ll \sum_{K_7<q\leq 2K_7} N_{2,a}(q) \ll K_2K_3K_5K_6K_7 + K_1K_2K_3K_4K_5K_6. \tag{4.15}$$

Alternatively, we may use $N_{2,b}(q)$, and write (4.3) as $\mathbf{v}.\mathbf{x} = 0$ with $\mathbf{v} = (n_2n_3, -n_5n_6, -q)$ and $\mathbf{x} = (n_1, n_4, n_7)$. We set $H_1 = 2K_1, H_2 = 2K_4$ as before, and $H_3 = 2K_7$. This time Lemma 6 produces a bound

$$\ll 1 + \frac{K_1K_4K_7}{\max H_i|v_i|} \ll 1 + \frac{K_1K_4K_7}{K_1K_2K_3}$$

for the number of triples (n_1, n_4, n_7). On summing over n_2, n_3, n_5, n_6 and q we then find that

$$N_2 \ll \sum_{q\ll K_8} N_{2,b}(q) \ll K_2K_3K_5K_6K_8 + K_4K_5K_6K_7K_8.$$

In view of (4.13) we may combine this with (4.15) to deduce that

$$N_2 \ll K_2K_3K_5K_6 \min(K_7, K_8) + K_1K_2K_3K_4K_5K_6.$$

Since we have

$$\min(K_7, K_8) \ll (K_1K_2K_3)^{1/2} \ll (K_1K_2K_3K_4K_5K_6)^{1/4},$$

by (4.10), we then deduce from (4.11) that

$$\begin{aligned} N_2 &\ll (K_1K_2K_3K_4K_5K_6)^{2/3}.(K_1K_2K_3K_4K_5K_6)^{1/4} \\ &\qquad\qquad + K_1K_2K_3K_4K_5K_6 \\ &\ll K_1K_2K_3K_4K_5K_6, \end{aligned}$$

which completes the proof of our bound for N_2.

We must now establish Lemma 5. To do this we refer to the author's work [10] on the divisor function $d_3(n)$ in arithmetic progressions. If we write

$N(K_1, K_2, K_3; a, q)$

$$= \#\{(n_1,n_2,n_3)\in\mathbb{N}^3 : K_i < n_i \le 2K_i,\ n_1n_2n_3 \equiv a \pmod{q}\}$$

then the analysis of [10, §7] suffices to show that

$$N(K_1,K_2,K_3;a,q) = C(K_1,K_2,K_3;q) + O(E),$$

with $C(K_1,K_2,K_3;q)$ independent of a, and an error term

$$E = (K_1K_2K_3)^{3/4+\varepsilon}q^{-13/24} + (K_1K_2K_3)^{7/9+\varepsilon}q^{-7/12} + (K_1K_2K_3)^{10/13+\varepsilon}q^{-15/26}$$
$$+(K_1K_2K_3)^{46/57+\varepsilon}q^{-12/19} + (K_1K_2K_3)^{86/107+\varepsilon}q^{-66/107},$$

for any fixed $\varepsilon > 0$. Since $q \ll (K_1K_2K_3)^{1/2}$ we deduce that

$$N(K_1,K_2,K_3;a,q) = C(K_1,K_2,K_3;q) + O(K_1K_2K_3q^{-1}).$$

We may now average over a coprime to q to find that

$$\phi(q)^{-1}\#\{(n_1,n_2,n_3)\in\mathbb{N}^3 : K_i < n_i \le 2K_i,\ \text{h.c.f.}(n_1n_2n_3,q)=1\}$$
$$= C(K_1,K_2,K_3;q) + O(K_1K_2K_3q^{-1}).$$

We deduce that $C(K_1,K_2,K_3;q) \ll K_1K_2K_3/\phi(q)$, and Lemma 5 follows. The reader should note that the work of Friedlander and Iwaniec [8] could have been used equally effectively at this point.

5 The Proof of Lemma 4

By symmetry, we may suppose at the outset that

$$K_1^2K_2K_3 \gg K_4^2K_5K_5. \tag{5.1}$$

We shall write $N_{3,a}(q)$ for the number of solutions $(n_1,\ldots,n_6)$ corresponding to each value $n_7 = q$, so that

$$N_3 \ll \sum_{K_7<q\le 2K_7} N_{3,a}(q). \tag{5.2}$$

Moreover, if we set

$$K_8 = K_1^2K_2K_3/K_7,$$

then it is apparent that we must have $K_8 \ll n_8 \ll K_8$ in any solution of (4.4). Thus

$$N_3 \ll \sum_{K_8\ll q\ll K_8} N_{3,b}(q), \tag{5.3}$$

where $N_{3,b}(q)$ counts the solutions corresponding to a given value $n_8 = q$.

We plan to use (5.2) when $K_7 \ll K_1(K_2K_3)^{1/2}$. If this condition fails to hold we must have $K_8 \ll K_1(K_2K_3)^{1/2}$, in which case we shall employ (5.3).

To bound $N_{3,a}(q)$ we write the condition (4.4) as $\mathbf{v}.\mathbf{x} = 0$ where $\mathbf{v} = (n_1^2n_2, n_4^2n_5, -q)$ and $\mathbf{x} = (n_3, n_6, n_8)$, say. We set $H_1 = 2K_3, H_2 = 2K_6$ and

$$H_3 = 32K_1^2K_2K_3/q.$$

Then Lemma 6 produces the bound $O(1+K_3K_6/q)$ for the number of triples (n_3, n_6, n_8) and it follows on summing over n_1, n_2, n_4 and n_5 that

$$N_{3,a}(q) \ll K_1K_2K_4K_5 + K_1K_2K_3K_4K_5K_6/q. \tag{5.4}$$

In a precisely analogous way we find that

$$N_{3,b}(q) \ll K_1K_3K_4K_6 + K_1K_2K_3K_4K_5K_6/q. \tag{5.5}$$

We may also use a vector $\mathbf{x}$ involving n_1 and n_4. To do this, we let $t \in [0, q)$ run over the solutions of the quadratic congruence

$$t^2n_2n_3 + n_5n_6 \equiv 0 \pmod{q},$$

and we write $\rho(q; n_2n_3, n_5n_6)$ for the number of such solutions t. We then see that, for fixed n_2, n_3, n_5, n_6 and q, we must have $n_1 \equiv tn_4 \pmod{q}$ for some value of t. This leads to an equation $\mathbf{v}.\mathbf{x} = 0$ with $\mathbf{v} = (1, -t, q)$ and $\mathbf{x} = (n_1, n_4, m)$, with size restrictions given by $H_1 = 2K_1, H_2 = 2K_4$ and $H_3 = K_1 + K_4$, say. Thus Lemma 6 produces a bound $O(1+K_1K_4/q)$ for the number of solutions n_1, n_4 corresponding to a given value of t. We therefore obtain an estimate

$$N_{3,a}(q) \ll (1 + K_1K_4/q) \sum_{n_2,n_3,n_5,n_6} \rho(q; n_2n_3, n_5n_6). \tag{5.6}$$

Our next task is evidently to examine averages of the function ρ. Suppose that h.c.f.$(ab, q) = 1$. Then for odd q we have

$$\rho(q; a, b) = \sum_{d|q} \mu(d)^2 (\frac{-ab}{d}),$$

where (ab/d) is the Jacobi symbol. We then see that

$$\rho(q; a, b) \leq 4 \sum_{d|q} \mu(d)^2 (\frac{-ab}{d})$$

whether q is even or odd, where we take the Jacobi symbol to vanish for even d. We also note that the sum on the right is non-negative when ab and q are not coprime. Our aim is to estimate

$$\sum_{q\le Q}\sum_{n_2,n_3,n_5,n_6}\rho(q;n_2n_3,n_5n_6)=S,$$

say. It will facilitate our argument to average over all 4-tuples (n_2,n_3,n_5,n_6) in the relevant ranges, and not just those satisfying the coprimality condition (4.1). In view of the above remarks we clearly have

$$\begin{aligned} S &\ll \sum_{e\le Q}\sum_{d\le Q/e}\mu(d)^2\sum_{n_2,n_3,n_5,n_6}(\frac{-n_2n_3n_5n_6}{d}) \\ &\ll KQ+\sum_{e\le Q}S(e), \end{aligned} \tag{5.7}$$

where

$$K=K_2K_3K_5K_6$$

and

$$S(e)=\sum_{1\ne d\le Q/e}\mu(d)^2\sum_{n_2,n_3,n_5,n_6}(\frac{-n_2n_3n_5n_6}{d}).$$

An immediate application of the author's large sieve inequality for real character sums [12, Corollary 4] shows that

$$S(e)\ll(KQ/e)^{\varepsilon}\{K(Q/e)^{1/2}+K^{1/2}(Q/e)\} \tag{5.8}$$

for any fixed $\varepsilon>0$. If we use the Pólya-Vinogradov inequality, we find that

$$\begin{aligned} S(e) &= \sum_{1\ne d\le Q/e}\mu(d)^2\sum_{n_2,n_3,n_5}(\frac{-n_2n_3n_5}{d})\sum_{n_6}(\frac{n_6}{d}) \\ &\ll K_2K_3K_5\sum_{1\ne d\le Q/e}d^{1/2+\varepsilon} \\ &\ll K_2K_3K_5(Q/e)^{3/2+\varepsilon}. \end{aligned}$$

In the same way we find that

$$S(e)\ll\frac{K_2K_3K_5K_6}{K_i}(Q/e)^{3/2+\varepsilon}$$

for any index $i=2,3,5,6$. It therefore follows on taking K_i as the maximum of K_2,K_3,K_5 and K_6, that

$$S(e)\ll K^{3/4}(Q/e)^{3/2+\varepsilon}. \tag{5.9}$$

Alternatively, if N is not a square, we may use the Pólya-Vinogradov to derive the bound

$$\begin{aligned}
\sum_{d\le Q/e}\mu(d)^2(\frac{N}{d}) &= \sum_{d\le Q/e}\sum_{h^2|d}\mu(h)(\frac{N}{d}) \\
&= \sum_{h\le (Q/e)^{1/2}}\mu(h)(\frac{N}{h^2})\sum_{k\le Q/eh^2}(\frac{N}{k}) \\
&\ll \sum_{h\le (Q/e)^{1/2}} N^{1/2}\log N \\
&\ll (QN/e)^{1/2}\log N. \qquad (5.10)
\end{aligned}$$

We can use this estimate to find that

$$S(e)\ll K^{3/2+\varepsilon}(Q/e)^{1/2}, \qquad (5.11)$$

since $-n_2n_3n_5n_6$ is never a square. Comparing this bound with (5.8) and (5.9) we find that

$$S(e)\ll (KQ/e)^{\varepsilon}\min,$$

where

$$\begin{aligned}
\min &= \min\{K(Q/e)^{1/2}+K^{1/2}(Q/e)\,,\,K^{3/4}(Q/e)^{3/2}\,,\,K^{3/2}(Q/e)^{1/2}\} \\
&\ll \min\{K(Q/e)^{1/2}\,,\,K^{3/4}(Q/e)^{3/2}\,,\,K^{3/2}(Q/e)^{1/2}\} \\
&\qquad +\min\{K^{1/2}(Q/e)\,,\,K^{3/4}(Q/e)^{3/2}\,,\,K^{3/2}(Q/e)^{1/2}\} \\
&\ll \{K(Q/e)^{1/2}\}^{3/5}\{K^{3/4}(Q/e)^{3/2}\}^{2/5} \\
&\qquad +\{K^{1/2}(Q/e)\}^{2/3}\{K^{3/2}(Q/e)^{1/2}\}^{1/3} \\
&\ll (KQ/e)^{9/10}+(KQ/e)^{5/6} \\
&\ll (KQ/e)^{9/10}.
\end{aligned}$$

It follows that

$$S(e)\ll (KQ/e)^{10/11}, \qquad (5.12)$$

say. Finally we insert this into (5.7) to deduce that

$$S\ll KQ+K^{10/11}Q^{10/11}\sum_{e\le Q}e^{-10/11}\ll KQ+K^{10/11}Q\ll KQ. \qquad (5.13)$$

The above bound allows us to conclude from (5.6) that

$$\sum_{Q/2<q\le Q}N_{3,a}(q)\ll (1+K_1K_4/Q)KQ=K_2K_3K_5K_6Q+K_1K_2K_3K_4K_5K_6.$$

On the other hand, (5.4) and (5.5) yield

$$\sum_{Q/2<q\le Q} N_{3,a}(q) \ll K_1K_2K_4K_5Q + K_1K_2K_3K_4K_5K_6$$

and

$$\sum_{Q/2<q\le Q} N_{3,a}(q) \ll K_1K_3K_4K_6Q + K_1K_2K_3K_4K_5K_6.$$

Taking the minimum of these, and assuming that $Q \ll K_1(K_2K_3)^{1/2}$, we obtain an estimate

$$\begin{aligned}
\sum_{Q/2<q\le Q} N_{3,a}(q) &\ll Q\min\{K_2K_3K_5K_6\,,\ K_1K_2K_4K_5\,,\ K_1K_3K_4K_6\} \\
&\qquad\qquad + K_1K_2K_3K_4K_5K_6 \\
&\ll Q\{K_2K_3K_5K_6\}^{1/2}\{K_1K_2K_4K_5\}^{1/4}\{K_1K_3K_4K_6\}^{1/4} \\
&\qquad\qquad + K_1K_2K_3K_4K_5K_6 \\
&\ll QK_1^{1/2}K_2^{3/4}K_3^{3/4}K_4^{1/2}K_5^{3/4}K_6^{3/4} + K_1K_2K_3K_4K_5K_6 \\
&\ll K_1(K_2K_3)^{1/2}.K_1^{1/2}K_2^{3/4}K_3^{3/4}K_4^{1/2}K_5^{3/4}K_6^{3/4} \\
&\qquad\qquad + K_1K_2K_3K_4K_5K_6 \\
&\ll K_1^{3/2}K_2^{5/4}K_3^{5/4}K_4^{1/2}K_5^{3/4}K_6^{3/4} \qquad (5.14)
\end{aligned}$$

in view of our assumption (5.1). This gives a satisfactory bound for (5.2). We may handle $N_{3,b}(q)$ in a precisely analogous way, thereby completing our treatment of (4.5).

The equation (4.6) introduces a couple of further difficulties. Firstly, the bound (5.10) is only valid when N is not a square. Previously we took $N = -n_2n_3n_5n_6$, which can never be a square. However, if $N = n_2n_3n_5n_6$, we must allow for the case in which $n_2n_3n_5n_6$ is a square. The effect of this is to change the estimate (5.11) into

$$S(e) \ll K^{3/2+\varepsilon}(Q/e)^{1/2} + K^{1/2+\varepsilon}(Q/e).$$

The additional term contributes $O(K^{2/3}Q/e)$, say, to (5.12), whence (5.13) becomes

$$S \ll KQ\{1 + \frac{\log(K_1K_2K_3K_4K_5K_6)}{K^{1/3}}\} \ll KQ\{1 + \frac{\log K_1K_4}{K^{1/3}}\}. \qquad (5.15)$$

This introduces the extra factor we see in (4.7).

The second difficulty is that if

$$K_4^2K_5K_6 \gg K_1^2K_2K_3 \gg K_4^2K_5K_6 \qquad (5.16)$$

we may no longer have the lower bound $q \gg K_8$ to use in the estimate

$$N_4 \ll \sum_{q \ll K_8} N_{4,b}(q). \tag{5.17}$$

We therefore assume now that (5.16) holds, and investigate the quantity $N_{4,b}(q)$ further. Since

$$n_1^2 n_2 n_3 - n_4^2 n_5 n_6 = n_7 q$$

in this context, with $n_7 \leq 2K_7$, we can apply Lemma 6 with

$$\mathbf{v} = (n_1^2 n_2, -n_4^2 n_5, q), \quad \mathbf{x} = (n_3, n_6, -n_7),$$

and with $H_1 = 2K_3, H_2 = 2K_6$ and $H_3 = 2K_7$. Thus there are

$$\ll 1 + \frac{H_1 H_2 H_3}{\max H_i |v_i|} \ll 1 + \frac{K_3 K_6 K_7}{K_1^2 K_2 K_3}$$

solutions $\mathbf{x}$. Summing over n_1, n_2, n_4, n_5 yields

$$N_{4,b}(q) \ll K_1 K_2 K_4 K_5 (1 + \frac{K_6 K_7}{K_1^2 K_2}). \tag{5.18}$$

Similarly one can show that

$$N_{4,b}(q) \ll K_1 K_3 K_4 K_6 (1 + \frac{K_5 K_7}{K_1^2 K_3}). \tag{5.19}$$

As before we need also an estimate in which we treat n_1 and n_4 as variables. By the argument used before we can produce $\rho(q; n_2 n_3, -n_5 n_6)$ congruence conditions $n_1 \equiv t n_4 (\mathrm{mod}\ q)$. Each of these defines a lattice $\Lambda \subseteq \mathbb{Z}^2$ of points (n_1, n_4). Moreover we will have $\det(\Lambda) = q$. The points (n_1, n_4) satisfy $n_1 \leq 2K_1$ and $n_4 \leq 2K_4$. Additionally we have

$$|1 - \frac{n_4^2 n_5 n_6}{n_1^2 n_2 n_3}| \leq \frac{2K_7 q}{K_2^2 K_2 K_3},$$

whence

$$|1 - \frac{n_4 \sqrt{n_5 n_6}}{n_1 \sqrt{n_2 n_3}}| \leq \frac{2K_7 q}{K_1^2 K_2 K_3}.$$

In view of our assumption (5.16) this may be written as

$$|n_1 - \alpha n_4| \leq C \frac{K_7 q}{K_1 K_2 K_3},$$

for some $\alpha = \alpha(n_2, n_3, n_4, n_5)$ and some absolute constant C. The above inequality, along with the condition $|n_4| \le 2K_4$, defines a parallelogram of area

$$8C\frac{K_4K_7q}{K_1K_2K_3} = A,$$

say, centred on the origin. This parallelogram may be mapped to a square S, centred on the origin, and having the same area A, by a projective mapping M say, of determinant 1. Enclose S by a disc D of area $\pi A/2$, and consider the ellipse $E = M^{-1}D$. This also has area $\pi A/2$. Moreover it contains the original parallelogram, and is centred at the origin. We are therefore in a position to apply the following result, due to the author [9, Lemma 2].

Lemma 7 *Let $\Lambda \subseteq \mathbb{R}^2$ be a lattice, and let E be an ellipse, centred on the origin, together with its interior. Then*

$$\#(\Lambda \cap E) \le 4\Big(1 + \frac{\mathrm{meas}(E)}{\det(\Lambda)}\Big).$$

This lemma allows us to conclude that there are

$$\ll 1 + \frac{K_4K_7}{K_1K_2K_3}$$

pairs (n_1, n_4) for each set of values t, n_2, n_3, n_5, n_6, q. We may now proceed as before, using (5.15) to deduce that

$$\sum_{q \ll K_8} N_{4,b}(q) \ll \tau K_2K_3K_5K_6\Big(1 + \frac{K_4K_7}{K_1K_2K_3}\Big)K_8, \tag{5.20}$$

where we have set

$$\tau = 1 + \frac{\log K_1K_4}{K^{1/3}}$$

for convenience. We now deduce from (5.17), (5.18), (5.19) and (5.20) that

$$\begin{aligned}
N_4 &\ll \sum_{q \ll K_8} N_{4,b}(q) \\
&\ll \tau \min(K_1K_2K_4K_5K_8\,,\ K_1K_3K_4K_6K_8\,,\ K_2K_3K_5K_6K_8) \\
&\qquad\qquad + \tau K_1K_2K_3K_4K_5K_6\frac{K_7K_8}{K_1^2K_2K_3} \\
&\ll \tau K_1^{1/2}K_2^{3/4}K_3^{3/4}K_4^{1/2}K_5^{3/4}K_6^{3/4}K_8 + \tau K_1K_2K_3K_4K_5K_6
\end{aligned}$$

as in the proof of (5.14). In the final line we have used the fact that $K_7K_8 \ll K_1^2K_2K_3$, as one sees from (4.6). Since we only need (5.17) for the case $K_8 \ll K_1(K_2K_3)^{1/2}$, the required bound (4.7) follows.

6 Proof of the Upper Bound

We shall specify dyadic ranges

$$X_i < |x_i| \le 2X_i$$

for the original variables x_1, x_2, x_3, x_4, and

$$Z_{ij} < z_{ij} \le 2Z_{ij}$$

for the variables introduced in §2, and we write

$$\mathcal{N}(X_1, \ldots, X_4; Z_{12}, \ldots, Z_{34}) = \mathcal{N}$$

for the corresponding contribution to $N(B)$. We obviously have

$$X_i \ll B, \tag{6.1}$$

and the relation (2.13) implies that

$$Z_{ij} Z_{ik} Z_{il} \ll X_i. \tag{6.2}$$

We shall find it convenient to re-order the indices so that

$$X_1 \ge X_2 \ge \max(X_3\,,\, X_4). \tag{6.3}$$

Since any solution will have

$$\frac{1}{2X_4} \le |\frac{1}{x_4}| = |\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}| \le |\frac{1}{x_1}| + |\frac{1}{x_2}| + |\frac{1}{x_3}| \le \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} \le \frac{3}{X_3},$$

we deduce that $\mathcal{N} = 0$ unless

$$6X_4 \ge X_3, \tag{6.4}$$

and similarly

$$6X_3 \ge X_4, \tag{6.5}$$

as we henceforth assume.

We also note that in any solution at least one inequality of the shape

$$|\frac{1}{x_1} + \frac{1}{x_i}| \ge \frac{1}{2|x_1|}, \quad (i = 2, 3, \text{ or } 4)$$

must hold, since otherwise we would have

$$|\frac{2}{x_1}| = |\{\frac{1}{x_1} + \frac{1}{x_2}\} + \{\frac{1}{x_1} + \frac{1}{x_3}\} + \{\frac{1}{x_1} + \frac{1}{x_4}\}| \le \frac{3}{2|x_1|},$$

giving a contradiction. Since we may interchange x_3 and x_4 without affecting the condition (6.3) it follows that we can divide our solutions into two classes, satisfying

$$|\frac{1}{x_1}+\frac{1}{x_h}|\geq\frac{1}{4X_1},\quad (h=2\text{ or }3) \tag{6.6}$$

respectively. We henceforth consider only solutions corresponding to a given value $h=2$ or 3. If $X_1\leq 4X_h$, then (6.6) yields

$$|\frac{1}{x_1}+\frac{1}{x_h}|\geq\frac{1}{16X_h}. \tag{6.7}$$

On the other hand, if $X_1\geq 4X_h$, then

$$|\frac{1}{x_1}+\frac{1}{x_h}|\geq\frac{1}{|x_h|}-\frac{1}{|x_1|}\geq\frac{1}{2X_h}-\frac{1}{X_1}\geq\frac{1}{2X_h}-\frac{1}{4X_h}=\frac{1}{4X_h}$$

and (6.7) is still true. Since we automatically have

$$|\frac{1}{x_1}+\frac{1}{x_h}|\leq\frac{1}{X_1}+\frac{1}{X_h}\leq\frac{2}{X_h}$$

we conclude that

$$\frac{1}{16X_h}\leq|\frac{1}{x_1}+\frac{1}{x_h}|\leq\frac{2}{X_h}. \tag{6.8}$$

We next observe that the relation (2.13) yields

$$\frac{x_jx_kx_lB_i}{(x_iA_i)^2}=\frac{x_1x_2x_3x_4P}{(x_iA_i)^3}=y_i^3,$$

with the notations (2.7), (2.12) and (3.3). Hence

$$Y_i\ll|y_i|\ll Y_i$$

where

$$Y_i=\frac{(XF)^{1/3}}{X_iZ_{jk}Z_{jl}Z_{kl}}, \tag{6.9}$$

and

$$X=X_1X_2X_3X_4\quad\text{and}\quad F=Z_{12}Z_{13}Z_{14}Z_{23}Z_{24}Z_{34}.$$

Moreover (2.16) implies

$$v_{12}=\frac{x_1x_2(\frac{1}{x_1}+\frac{1}{x_2})}{z_{12}^2y_3y_4},$$

and similarly for v_{13}. Thus, in view of (6.8), we see that

$$W_h \ll |v_{1h}| \ll W_h \tag{6.10}$$

for some function W_h of the various X_i and Z_{ij}.

After these preliminaries, we begin by applying Lemma 3 to the equation (2.16), to show that there are

$$\ll Z_{ik}Z_{il}Z_{jk}Z_{jl}Y_iY_j$$

possible sets of values for $z_{ik}, z_{il}, z_{jk}, z_{jl}, z_{ij}, y_i, y_j, v_{ij}$. For each set of values we proceed to examine (2.17), which we write in the form $\mathbf{v}.\mathbf{x} = 0$ with

$$\mathbf{v} = (v_{ij}, -z_{il}^2 y_j, z_{jk}^2 y_i) \quad \text{and} \quad \mathbf{x} = (v_{ik}, y_k, y_l).$$

In view of (2.2), (2.10) and (2.15) both $\mathbf{v}$ and $\mathbf{x}$ will be primitive. Moreover, the equation

$$z_{ij}z_{il}y_k + z_{jk}z_{kl}y_i = z_{ik}v_{ik},$$

which is an example of (2.16), yields

$$z_{ik}v_{ik} \ll \max\{Z_{ij}Z_{il}Y_k\,,\ Z_{jk}Z_{kl}Y_i\} = \frac{(XF)^{1/3}}{Z_{jl}}\max\{\frac{1}{X_k}\,,\ \frac{1}{X_i}\}.$$

Thus Lemma 6 may be applied with

$$H_1 = c\frac{(XF)^{1/3}}{Z_{ik}Z_{jl}}\max\{\frac{1}{X_k}\,,\ \frac{1}{X_i}\}, \quad H_2 = cY_k, \quad H_3 = cY_l,$$

for a suitable constant c. Since the remaining value z_{kl} is determined by (2.14), there are

$$\mathcal{N} \ll Z_{ik}Z_{il}Z_{jk}Z_{jl}Y_iY_j\{1 + \frac{H_1H_2H_3}{\max(H_2V_2\,,\ H_3V_3)}\}$$

solutions to (2.14) in total, where

$$V_2 = Z_{il}^2Y_j \quad \text{and} \quad V_3 = Z_{jk}^2Y_i.$$

We may then calculate, using (6.9), that the above bound is

$$\ll \frac{(XF)^{2/3}}{Z_{kl}^2X_iX_j} + \max(\frac{1}{X_k}\,,\ \frac{1}{X_i})\min(X_iX_l\,,\ X_jX_k).$$

If we take $\{i,k\} = \{1,2\}$ then $\min(X_iX_l, X_jX_k) \le X_2X_3$ and

$$\max(\frac{1}{X_k}\,,\ \frac{1}{X_i}) \le \frac{1}{X_2},$$

by (6.3). If we choose the indices i, k in this way, it will follow that

$$\mathcal{N}(X_1,\ldots,X_4;Z_{12},\ldots,Z_{34}) \ll \frac{(XF)^{2/3}}{Z_{kl}^2 X_i X_j} + X_3.$$

We apply this with $i=1,\ j=4,\ k=2,\ l=3$, so that

$$\mathcal{N}(X_1,\ldots,X_4;Z_{12},\ldots,Z_{34}) \ll \frac{(XF)^{2/3}}{Z_{23}^2 X_1 X_4} + X_3,$$

and again with $i=2,\ j=3,\ k=1,\ l=4$, so that

$$\mathcal{N}(X_1,\ldots,X_4;Z_{12},\ldots,Z_{34}) \ll \frac{(XF)^{2/3}}{Z_{14}^2 X_2 X_3} + X_3.$$

Since

$$\min(A,B) \le (AB)^{1/2} \tag{6.11}$$

this yields

$$\begin{aligned}\mathcal{N}(X_1,\ldots,X_4;Z_{12},\ldots,Z_{34}) &\ll \min\{\frac{(XF)^{2/3}}{Z_{23}^2 X_1 X_4}, \frac{(XF)^{2/3}}{Z_{14}^2 X_2 X_3}\} + X_3\\ &\ll \frac{(XF)^{2/3}}{Z_{14}Z_{23}X^{1/2}} + X_3\\ &= \frac{X^{1/6}F^{2/3}}{Z_{14}Z_{23}} + X_3. \end{aligned} \tag{6.12}$$

For an alternative estimate we begin by applying Lemma 4 to the equation (2.17), putting $i=1,\ j=3,\ k=2,$ and $l=4$. Thus we may either take v_{12} or v_{13} to be n_7, depending on the value of h, the necessary constraints coming from (6.10). We then see that the number of possible sets of values for $v_{12}, v_{13}, z_{14},\ z_{23}, y_1, y_3, y_2, y_4$ is

$$\begin{aligned} &\ll \{1+\frac{\log X}{(Y_1Y_2Y_3Y_4)^{1/3}}\}Z_{14}Z_{23}Y_1Y_2Y_3Y_4\max\{(\frac{Z_{14}^2Y_2Y_3}{Z_{23}^2Y_1Y_4})^{1/4}, (\frac{Z_{23}^2Y_1Y_4}{Z_{14}^2Y_2Y_3})^{1/4}\}\\ &\ll \sigma Z_{14}Z_{23}X^{1/3}F^{-2/3}\max\{(\frac{X_1X_4}{X_2X_3})^{1/4}, (\frac{X_2X_3}{X_1X_4})^{1/4}\}\\ &\ll \sigma Z_{14}Z_{23}X^{1/3}F^{-2/3}(X_1/X_2)^{1/4},\end{aligned}$$

by (6.3), (6.4) and (6.5), where

$$\sigma = 1+\frac{\log X}{(XF^{-2})^{1/9}}.$$

For each such set of values we write (2.16) in the form $\mathbf{v}.\mathbf{x} = 0$ with

$$\mathbf{v} = (z_{14}y_3, z_{23}y_1, -v_{13}) \quad \text{and} \quad \mathbf{x} = (z_{12}, z_{34}, z_{13}).$$

In view of (2.2), (2.10) and (2.15) both $\mathbf{v}$ and $\mathbf{x}$ will be primitive. We can therefore apply Lemma 6 with

$$H_1 = 2Z_{12}, \quad H_2 = 2Z_{34}, \quad H_3 = 2Z_{13},$$

to deduce that there are

$$\begin{aligned}
&\ll \; 1 + \frac{H_1H_2H_3}{\max(H_1|v_1|\,,\, H_2|v_2|)} \\
&\ll \; 1 + \frac{Z_{12}Z_{34}Z_{13}}{\max(Z_{12}Z_{14}Y_3\,,\, Z_{23}Z_{34}Y_1)} \\
&\ll \; 1 + \frac{Z_{12}Z_{34}Z_{13}Z_{24}}{(XF)^{1/3}} \min(X_1\,,\, X_3) \\
&\ll \; 1 + \frac{Z_{12}Z_{34}Z_{13}Z_{24}}{(XF)^{1/3}} X_3
\end{aligned}$$

corresponding solutions z_{12}, z_{34}, z_{13}. Since the final remaining value $z_{kl} = z_{24}$ is now determined by (2.14) it follows that

$$\begin{aligned}
\mathcal{N} &\ll \sigma Z_{14}Z_{23}X^{1/3}F^{-2/3}(X_1/X_2)^{1/4}\left(1 + \frac{Z_{12}Z_{34}Z_{13}Z_{24}}{(XF)^{1/3}}X_3\right) \\
&= \sigma Z_{14}Z_{23}X^{1/3}F^{-2/3}(X_1/X_2)^{1/4} + \sigma(X_1/X_2)^{1/4}X_3.
\end{aligned}$$

We now combine this with (6.12), using the inequality (6.11) again, to deduce that

$$\begin{aligned}
\mathcal{N} &\ll \sigma \min\{\frac{X^{1/6}F^{2/3}}{Z_{14}Z_{23}}\,,\, Z_{14}Z_{23}X^{1/3}F^{-2/3}(X_1/X_2)^{1/4}\} + \sigma(X_1/X_2)^{1/4}X_3 \\
&\ll \sigma X^{1/4}(X_1/X_2)^{1/8} + \sigma(X_1/X_2)^{1/4}X_3.
\end{aligned}$$

We are finally in a position to sum over the various dyadic ranges for the X_i and Z_{ij}, subject to (6.1) and (6.2). We begin by considering the summation over Z_{ij}. The values of Z_{ij} are powers of 2, subject to the constraints (6.2). These imply that $F^2 \ll X \ll B^4$. Thus there are $O((\log B)^6)$ possible sets of values for the various Z_{ij}. Moreover there are $O((\log B)^5)$ sets of values for each given value of F. Since F runs over powers of 2, subject to $F \ll X^{1/2}$ we conclude that

$$\sum_{Z_{ij}} F^{2/9} \ll X^{1/9}(\log B)^5.$$

We therefore deduce that

$$\sum_{Z_{ij}} \sigma \ll (\log B)^6. \tag{6.13}$$

It remains to consider the summation over values of the X_i, which also run over powers of 2. Here we observe that

$$X^{1/4}(X_1/X_2)^{1/8} = X_1^{3/8} X_2^{1/8} X_3^{1/4} X_4^{1/4},$$

and that (6.1) yields

$$\sum_{X_i} X_i^e \ll B^e$$

if $e > 0$. It follows that

$$\sum_{X_1,X_2,X_3,X_4} X_1^{3/8} X_2^{1/8} X_3^{1/4} X_4^{1/4} \ll B.$$

Similarly we have

$$(X_1/X_2)^{1/4} X_3 \ll X_1^{1/4} X_2^{1/4} X_3^{1/4} X_4^{1/4},$$

by (6.3) and (6.4), so that

$$\sum_{X_1,X_2,X_3,X_4} (X_1/X_2)^{1/4} X_3 \ll B.$$

Combining this with (6.13) completes the proof of the upper bound in our theorem.

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Mathematical Institute,
24-29, St. Giles',
Oxford OX1 3LB
England

email: rhb@maths.ox.ac.uk